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Bounded Rationality in Concurrent Parity Games

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Abstract

We consider 2-player games played on a finite state space for an infinite number of rounds. The games are *concurrent*: in each round, the two players (player 1 and player 2) choose their moves independently and simultaneously; the current state and the two moves determine the successor state. We study concurrent games with ω-regular winning conditions specified as *parity* objectives. We consider the *qualitative analysis* problems: the computation of the *almost-sure* and *limit-sure* winning set of states, where player 1 can ensure to win with probability 1 and with probability arbitrarily close to 1, respectively. In general the almost-sure and limit-sure winning strategies require both *infinite-memory* as well as *infinite-precision* (to describe probabilities). We study the *bounded-rationality* problem for qualitative analysis of concurrent parity games, where the strategy set for player 1 is restricted to bounded-resource strategies. In terms of precision, strategies can be deterministic, uniform, finite-precision or infinite-precision; and in terms of memory, strategies can be memoryless, finite-memory or infinite-memory. We present a precise and complete characterization of the qualitative winning sets for all combinations of classes of strategies. In particular, we show that uniform memoryless strategies are as powerful as finite-precision infinite-memory strategies, and infinite-precision memoryless strategies are as powerful as infinite-precision finite-memory strategies. We show that the winning sets can be computed in $\mathcal{O}(n^{2d+3})$ time, where n is the size of the game structure and $2d$ is the number of priorities (or colors), and our algorithms are symbolic. The membership problem of whether a state belongs to a winning set can be decided in NP $\cap$ coNP. While this complexity is the same as for the simpler class of *turn-based* parity games, where in each state only one of the two players has a choice of moves, our algorithms, that are obtained by characterization of the winning sets as μ-calculus formulas, are considerably more involved than those for turn-based games.

1 Introduction

Concurrent games are played by two players on a finite state space for an infinite number of rounds. In each round, the two players independently choose moves, and the current state and the two chosen moves determine the successor state. In *deterministic* concurrent games, the successor state is unique; in *probabilistic* concurrent games, the successor state is given by a probability distribution. The outcome of the game (or a *play*) is an infinite sequence of states. These games were introduced by Shapley [Sha53], and has been one of the most fundamental and well studied game models in stochastic graph games. We consider ω-regular objectives; that is, given an ω-regular set Φ of infinite state sequences, player 1 wins if the outcome of the game lies in Φ. Otherwise, player 2 wins, i.e., the game is zero-sum. Such games occur in the synthesis and verification of reactive systems [Chu62, RW87, PR89] (see also [ALW89, Dil89, AHK97]).

The player-1 *value* $v_1(s)$ of the game at a state s is the limit probability with which player 1 can ensure that the outcome of the game lies in Φ; that is, the value $v_1(s)$ is the maximal probability with which player 1 can guarantee Φ against all strategies of player 2. Symmetrically, the player-2 *value* $v_2(s)$ is the limit probability with which player 2 can ensure that the outcome of the game lies outside Φ. The *qualitative* analysis of games asks for the computation of the set of *almost-sure* winning states where player 1 can ensure Φ with probability 1, and the set of *limit-sure* winning states where player 1 can ensure Φ with probability arbitrarily close to 1 (states with value 1); and the *quantitative* analysis asks for a precise computation of values.

Traditionally, the special case of *turn-based* games has received most attention. In turn-based games, in each round, only one of the two players has a choice of moves. In turn-based deterministic games, all values are 0 or 1 and can be computed using combinatorial algorithms [Tho90, Sch07, JPZ06]; in turn-based probabilistic games, values can be computed by iterative approximation [CH06, Con92, GH08]. In this paper we focus on the more general *concurrent* situation, where in each round, both players choose their moves simultaneously and independently. Such concurrency is necessary for modeling the synchronous interaction of components [dAHM00, dAHM01]. The concurrent probabilistic games fall into a class of stochastic games studied in game theory [Sha53], and the ω-regular objectives, which arise from the safety and liveness specifications of reactive systems, fall into a low level ($\Sigma_3 \cap \Pi_3$) of the Borel hierarchy. From a classical result of Martin [Mar98] that established determinacy of Blackwell games it follows that concurrent probabilistic ω-regular games are determined, i.e., for each state s we have $v_1(s) + v_2(s) = 1$. Parity objectives can express all ω-regular conditions, and we consider concurrent games with parity objectives.

Concurrent games differ from turn-based games in that optimal strategies require, in general, randomization. A *pure* strategy must, in each round, choose a move based on the current state and the history (i.e., past state sequence) of the game. By contrast, a *randomized* strategy in each round chooses a probability distribution over moves (rather than a single move). The move to be played is then selected at random, according to the chosen distribution. Randomized strategies are not helpful for achieving a value of 1 in turn-based probabilistic games [CJH03, Cha07], but they can be helpful in concurrent games, even if the game itself is deterministic [dAHK07]. In contrast to turn-based deterministic and probabilistic games with parity objectives, where deterministic memoryless strategies exist for qualitative analysis [EJ88, Zie98, DJW97, CJH03, Cha07], in concurrent games, along with randomization, infinite-memory is required for limit-sure winning [dAH00].

The strategies for qualitative analysis for concurrent games require two different types of infinite resource: (a) infinite-memory, and (b) infinite-precision in describing the probabilities in the randomized strategies; (see example in [dAH00] that limit-sure winning in concurrent Büchi games require both infinite-memory and infinite-precision). In many applications, such as synthesis of reactive systems, infinite-memory and infinite-precision strategies are not implementable in practice. Thus though the theoretical solution of infinite-memory and infinite-precision strategies was established in [dAH00], the strategies obtained are not realizable in practice, and the theory to obtain implementable strategies in such games has not been studied before. In this work we consider the *bounded rationality* problem for qualitative analysis of concurrent parity games, where player 1 (that represents the controller) can play strategies with bounded resource. To the best of our knowledge this is the first work that considers the bounded rationality problem for concurrent ω-regular graph games. The motivation is clear as controllers obtained from infinite-memory and infinite-precision strategies are not implementable.

In terms of precision, strategies can be classified as pure (deterministic), uniformly random, finite-precision, and infinite-precision (in increasing order of precision to describe probabilities of a randomized strategy). In terms of memory, strategies can be classified as memoryless, finite-memory and infinite-memory. In [dAH00] the almost-sure and limit-sure winning characterization under infinite-memory, infinite-precision strategies were presented. In this work, we present (i) a complete and precise characterization of the qualitative winning sets for bounded resource strategies, (ii) symbolic algorithms to compute the winning sets, and (iii) complexity results to determine whether a given state belongs to a qualitative winning set.

Our contributions for bounded rationality in concurrent parity games are summarized below.

1. We show that pure memoryless strategies are as powerful as pure infinite-memory strategies. This result is obtained by a simple reduction to turn-based stochastic games.

2. We show that uniform memoryless strategies are more powerful than pure infinite-memory strategies, and uniform memoryless strategies are as powerful as finite-precision infinite-memory strategies. Thus our results show that if player 1 has only finite-precision strategies, then no memory is required and uniform randomization is sufficient. Hence very simple (uniform memoryless) controllers can be obtained for the entire class of finite-precision infinite-memory controllers. The result is obtained by a reduction to turn-based stochastic games, and the main technical contribution is the characterization of the winning sets for uniform memoryless strategies by a μ-calculus formula. The μ-calculus formula not only gives a symbolic algorithm, but is also in the heart of other proofs of the paper. The μ-calculus formula and the correctness proof are non-trivial generalizations of the classical result of Emerson-Jutla [EJ91] for turn-based deterministic parity games.

3. In case of finite-precision strategies, the almost-sure and limit-sure winning sets coincide. For almost-sure winning, uniform memoryless strategies are also as powerful as infinite-precision finite-memory strategies. However, we show with an example that infinite-memory infinite-precision strategies are more powerful than uniform memoryless strategies for almost-sure winning. For limit-sure winning, we show that infinite-precision memoryless strategies are more powerful than finite-precision infinite-memory strategies, and infinite-precision memoryless strategies are as powerful as infinite-precision finite-memory strategies. Our results show that if infinite-memory is not available, then no memory is required (memoryless strategies are as powerful as finite-memory strategies). The result is obtained by using the μ-calculus formula for the uniform memoryless case: we show that a μ-calculus formula that combines the μ-calculus formula for almost-sure winning for uniform memoryless strategies and limit-sure winning for reachability with memoryless strategies exactly characterizes the limit-sure winning for parity objectives for memoryless strategies.

4. As a consequence of the characterization of the winning sets as μ-calculus formulas we obtain symbolic algorithms to compute the winning sets. We show that the winning sets can be computed in $\mathcal{O}(n^{2d+3})$ time, where n is the size of the game structure and $2d$ is the number of priorities (or colors), and our algorithms are symbolic.

5. The membership problem of whether a state belongs to a winning set can be decided in NP $\cap$ coNP. While this complexity is the same as for the simpler class of *turn-based* parity games, where in each state only one of the two players has a choice of moves, our algorithms, that are obtained by characterization of the winning sets as μ-calculus formulas, are considerably more involved than those for turn-based games.

In short, our results show that if infinite-memory is not available, then memory is useless, and if infinite-precision is not available, then uniform memoryless strategies are sufficient. Let P, U, FP, IP denote pure, uniform, finite-precision, and infinite-precision strategies, respectively, and M, FM, IM denote memoryless, finite-memory, and infinite-memory strategies, respectively. For $A \in \{P, U, FP, IP\}$ and $B \in \{M, FM, IM\}$, let $\mathit{Almost}_1(A, B, \Phi)$ denote the almost-sure winning set under player 1 strategies that are restricted to be both A and B for a parity objective Φ (and similar notation for $\mathit{Limit}_1(A, B, \Phi)$). Then our results can be summarized by the following equalities and strict inclusion:

$$\begin{array}{rcl} \mathit{Almost}_1(P, M, \Phi) & = & \mathit{Almost}_1(P, IM, \Phi) = \mathit{Limit}_1(P, IM, \Phi) \\ & \subsetneq & \mathit{Almost}_1(U, M, \Phi) = \mathit{Almost}_1(FP, IM, \Phi) \\ & = & \mathit{Limit}_1(FP, IM, \Phi) = \mathit{Almost}_1(IP, FM, \Phi) \subsetneq \mathit{Almost}_1(IP, IM, \Phi). \end{array}$$

$$\mathit{Limit}_1(FP, IM, \Phi) \subsetneq \mathit{Limit}_1(IP, M, \Phi) = \mathit{Limit}_1(IP, FM, \Phi) \subsetneq \mathit{Limit}_1(IP, IM, \Phi).$$

2 Definitions

In this section we define game structures, strategies, objectives, winning modes and give other preliminary definitions.

2.1 Game structures

Probability distributions. For a finite set A, a *probability distribution* on A is a function $\delta : A \mapsto [0, 1]$ such that $\sum_{a \in A} \delta(a) = 1$. We denote the set of probability distributions on A by $\mathcal{D}(A)$. Given a distribution $\delta \in \mathcal{D}(A)$, we denote by $\mathrm{Supp}(\delta) = \{x \in A \mid \delta(x) > 0\}$ the *support* of the distribution δ.

Concurrent game structures. A (two-player) *concurrent stochastic game structure* $\mathcal{G} = \langle S, A, \Gamma_1, \Gamma_2, \delta \rangle$ consists of the following components.

- A finite state space S.
- A finite set A of moves (or actions).
- Two move assignments $\Gamma_1, \Gamma_2 : S \mapsto 2^A \setminus \emptyset$. For $i \in \{1, 2\}$, assignment Γ_i associates with each state $s \in S$ the nonempty set $\Gamma_i(s) \subseteq A$ of moves available to player i at state s. For technical convenience, we assume that $\Gamma_i(s) \cap \Gamma_j(t) = \emptyset$ unless $i = j$ and $s = t$, for all $i, j \in \{1, 2\}$ and $s, t \in S$. If this assumption is not met, then the moves can be trivially renamed to satisfy the assumption.
- A probabilistic transition function $\delta : S \times A \times A \mapsto \mathcal{D}(S)$, which associates with every state $s \in S$ and moves $a_1 \in \Gamma_1(s)$ and $a_2 \in \Gamma_2(s)$ a probability distribution $\delta(s, a_1, a_2) \in \mathcal{D}(S)$ for the successor state.

Plays. At every state $s \in S$, player 1 chooses a move $a_1 \in \Gamma_1(s)$, and simultaneously and independently player 2 chooses a move $a_2 \in \Gamma_2(s)$. The game then proceeds to the successor state t with probability $\delta(s, a_1, a_2)(t)$, for all $t \in S$. For all states $s \in S$ and moves $a_1 \in \Gamma_1(s)$ and $a_2 \in \Gamma_2(s)$, we indicate by $Dest(s, a_1, a_2) = \mathrm{Supp}(\delta(s, a_1, a_2))$ the set of possible successors of s when moves a_1, a_2 are selected. A *path* or a *play* of $\mathcal{G}$ is an infinite sequence $\omega = \langle s_0, s_1, s_2, \ldots \rangle$ of states in S such that for all $k \geq 0$, there are moves $a_1^k \in \Gamma_1(s_k)$ and $a_2^k \in \Gamma_2(s_k)$ such that $s_{k+1} \in Dest(s_k, a_1^k, a_2^k)$. We denote by Ω the set of all paths. For a play $\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega$, we define $\mathit{Inf}(\omega) = \{s \in S \mid s_k = s \text{ for infinitely many } k \geq 0\}$ to be the set of states that occur infinitely often in ω.

Size of a game. The *size* of a concurrent game is the sum of the size of the state space and the number of the entries of the transition function. Formally the size of a game is $|S| + \sum_{s \in S, a \in \Gamma_1(s), b \in \Gamma_2(s)} |Dest(s, a, b)|$.

Turn-based stochastic games and MDPs. A game structure $\mathcal{G}$ is *turn-based stochastic* if at every state at most one player can choose among multiple moves; that is, for every state $s \in S$ there exists at most one $i \in \{1, 2\}$ with $|\Gamma_i(s)| > 1$. A game structure is a player-2 *Markov decision process* if for all $s \in S$ we have $|\Gamma_1(s)| = 1$, i.e., only player-2 has choice of actions in the game.

Equivalent game structures. Given two game structures $\mathcal{G}_1 = \langle S, A, \Gamma_1, \Gamma_2, \delta_1 \rangle$ and $\mathcal{G}_2 = \langle S, A, \Gamma_1, \Gamma_2, \delta_2 \rangle$ on the same state and action space, with different transition function, we say that $\mathcal{G}_1$ is equivalent to $\mathcal{G}_2$ (denoted $\mathcal{G}_1 \equiv \mathcal{G}_2$) if for all $s \in S$ and all $a_1 \in \Gamma_1(s)$ and $a_2 \in \Gamma_2(s)$ we have $\text{Supp}(\delta_1(s, a_1, a_2)) = \text{Supp}(\delta_2(s, a_1, a_2))$.

2.2 Strategies

A *strategy* for a player is a recipe that describes how to extend a play. Formally, a strategy for player $i \in \{1, 2\}$ is a mapping $\pi_i : S^+ \mapsto \mathcal{D}(A)$ that associates with every nonempty finite sequence $x \in S^+$ of states, representing the past history of the game, a probability distribution $\pi_i(x)$ used to select the next move. The strategy π_i can prescribe only moves that are available to player i; that is, for all sequences $x \in S^*$ and states $s \in S$, we require that $\text{Supp}(\pi_i(x \cdot s)) \subseteq \Gamma_i(s)$. We denote by Π_i the set of all strategies for player $i \in \{1, 2\}$.

Given a state $s \in S$ and two strategies $\pi_1 \in \Pi_1$ and $\pi_2 \in \Pi_2$, we define *Outcomes*$(s, \pi_1, \pi_2) \subseteq \Omega$ to be the set of paths that can be followed by the game, when the game starts from s and the players use the strategies π_1 and π_2. Formally, $\langle s_0, s_1, s_2, \ldots \rangle \in$ *Outcomes*(s, π_1, π_2) if $s_0 = s$ and if for all $k \geq 0$ there exist moves $a_1^k \in \Gamma_1(s_k)$ and $a_2^k \in \Gamma_2(s_k)$ such that

$$\pi_1(s_0, \ldots, s_k)(a_1^k) > 0, \qquad \pi_2(s_0, \ldots, s_k)(a_2^k) > 0, \qquad s_{k+1} \in Dest(s_k, a_1^k, a_2^k).$$

Once the starting state s and the strategies π_1 and π_2 for the two players have been chosen, the probabilities of events are uniquely defined [Var85], where an *event* $\mathcal{A} \subseteq \Omega$ is a measurable set of paths[1]. For an event $\mathcal{A} \subseteq \Omega$, we denote by $\text{Pr}_s^{\pi_1, \pi_2}(\mathcal{A})$ the probability that a path belongs to $\mathcal{A}$ when the game starts from s and the players use the strategies π_1 and π_2.

Classification of strategies. We classify strategies according to their use of *randomization* and *memory*. We first present the classification according to randomization.

1. *(Pure).* A strategy π is *pure (deterministic)* if for all $x \in S^+$ there exists $a \in A$ such that $\pi(x)(a) = 1$. Thus, deterministic strategies are equivalent to functions $S^+ \mapsto A$.

2. *(Uniform).* A strategy π is *uniform* if for all $x \in S^+$ we have $\pi(x)$ is uniform over its support, i.e., for all $a \in \text{Supp}(\pi(x))$ we have $\pi(x)(a) = \frac{1}{|\text{Supp}(\pi(x))|}$.

3. *(Finite-precision).* A strategy π is *finite-precision* if there exists a bound $b \in \mathbb{N}$ such that for all $x \in S^+$ and all actions a we have $\pi(x)(a) = \frac{i}{j}$, where $i, j \in \mathbb{N}$ and $0 \leq i \leq j \leq b$ and $j > 0$, i.e., the probability of an action played by the strategy is a multiple of some $\ell \in \mathbb{N}$ such that $\ell \leq b$.

We denote by $\Pi_i^P, \Pi_i^U, \Pi_i^{FP}$ and Π_i^{IP} the set of pure (deterministic), uniform, finite-precision, and infinite-precision (or general) strategies for player i, respectively. Observe that we have the following strict inclusion: $\Pi_i^P \subset \Pi_i^U \subset \Pi_i^{FP} \subset \Pi_i^{IP}$.

[1] To be precise, we should define events as measurable sets of paths *sharing the same initial state*, and we should replace our events with families of events, indexed by their initial state [KSK66]. However, our (slightly) improper definition leads to more concise notation.

1. *(Finite-memory).* Strategies in general are *history-dependent* and can be represented as follows: let $\mathcal{M}$ be a set called *memory* to remember the history of plays (the set $\mathcal{M}$ can be infinite in general). A strategy with memory can be described as a pair of functions: (a) a *memory update* function $\pi_u : S \times \mathcal{M} \mapsto \mathcal{M}$, that given the memory $\mathcal{M}$ with the information about the history and the current state updates the memory; and (b) a *next move* function $\pi_n : S \times \mathcal{M} \mapsto \mathcal{D}(A)$ that given the memory and the current state specifies the next move of the player. A strategy is *finite-memory* if the memory $\mathcal{M}$ is finite.

2. *(Memoryless).* A *memoryless* strategy is independent of the history of play and only depends on the current state. Formally, for a memoryless strategy π we have $\pi(x \cdot s) = \pi(s)$ for all $s \in S$ and all $x \in S^*$. Thus memoryless strategies are equivalent to functions $S \mapsto \mathcal{D}(A)$.

We denote by Π_i^M, Π_i^{FM} and Π_i^{IM} the set of memoryless, finite-memory, and infinite-memory (or general) strategies for player i, respectively. Observe that we have the following strict inclusion: $\Pi_i^M \subset \Pi_i^{FM} \subset \Pi_i^{IM}$.

2.3 Objectives

We specify objectives for the players by providing the set of *winning plays* $\Phi \subseteq \Omega$ for each player. In this paper we study only zero-sum games [RF91, FV97], where the objectives of the two players are complementary. A general class of objectives are the Borel objectives [Kec95]. A *Borel objective* $\Phi \subseteq S^\omega$ is a Borel set in the Cantor topology on S^ω. In this paper we consider ω*-regular objectives* [Tho90], which lie in the first $2^1/_2$ levels of the Borel hierarchy (i.e., in the intersection of Σ_3 and Π_3). We will consider the following ω-regular objectives.

- *Reachability and safety objectives.* Given a set $T \subseteq S$ of "target" states, the reachability objective requires that some state of T be visited. The set of winning plays is thus $\text{Reach}(T) = \{\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega \mid \exists k \geq 0.\ s_k \in T\}$. Given a set $F \subseteq S$, the safety objective requires that only states of F be visited. Thus, the set of winning plays is $\text{Safe}(F) = \{\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega \mid \forall k \geq 0.\ s_k \in F\}$.

- *Büchi and co-Büchi objectives.* Given a set $B \subseteq S$ of "Büchi" states, the Büchi objective requires that B is visited infinitely often. Formally, the set of winning plays is $\text{Büchi}(B) = \{\omega \in \Omega \mid \textit{Inf}(\omega) \cap B \neq \emptyset\}$. Given $C \subseteq S$, the co-Büchi objective requires that all states visited infinitely often are in C. Formally, the set of winning plays is $\text{co-Büchi}(C) = \{\omega \in \Omega \mid \textit{Inf}(\omega) \subseteq C\}$.

- *Parity objectives.* For $c, d \in \mathbb{N}$, we let $[c..d] = \{c, c+1, \ldots, d\}$. Let $p : S \mapsto [0..d]$ be a function that assigns a *priority* $p(s)$ to every state $s \in S$, where $d \in \mathbb{N}$. The *Even parity objective* requires that the maximum priority visited infinitely often is even. Formally, the set of winning plays is defined as $\text{Parity}(p) = \{\omega \in \Omega \mid \max\big(p(\textit{Inf}(\omega))\big) \text{ is even }\}$. The dual *Odd parity objective* is defined as $\text{coParity}(p) = \{\omega \in \Omega \mid \max\big(p(\textit{Inf}(\omega))\big) \text{ is odd }\}$. Note that for a priority function $p : S \mapsto \{1, 2\}$, an even parity objective $\text{Parity}(p)$ is equivalent to the Büchi objective $\text{Büchi}(p^{-1}(2))$, i.e., the Büchi set consists of the states with priority 2. Hence Büchi and co-Büchi objectives are simpler and special cases of parity objectives.

Given a set $U \subseteq S$ we use usual LTL notations $\Box U, \Diamond U, \Box\Diamond U$ and $\Diamond\Box U$ to denote $\text{Safe}(U), \text{Reach}(U), \text{Büchi}(U)$ and $\text{co-Büchi}(U)$, respectively. Parity objectives are of special importance as they can express all ω-regular objectives, and hence all commonly used specifications in verification [Tho90].

2.4 Winning modes

Given an objective Φ, for all initial states $s \in S$, the set of paths Φ is measurable for all choices of the strategies of the player [Var85]. Given an initial state $s \in S$, an objective Φ, and a class $\Pi_1^{\mathcal{C}}$ of strategies we consider the following *winning modes* for player 1:

Almost. We say that player 1 *wins almost surely* with the class $\Pi_1^{\mathcal{C}}$ if the player has a strategy in $\Pi_1^{\mathcal{C}}$ to win with probability 1, or $\exists \pi_1 \in \Pi_1^{\mathcal{C}} . \forall \pi_2 \in \Pi_2 . \Pr_s^{\pi_1,\pi_2}(\Phi) = 1$.

Limit. We say that player 1 *wins limit surely* with the class $\Pi_1^{\mathcal{C}}$ if the player can ensure to win with probability arbitrarily close to 1 with $\Pi_1^{\mathcal{C}}$, in other words, for all $\varepsilon > 0$ there is a strategy for player 1 in $\Pi_1^{\mathcal{C}}$ that ensures to win with probability at least $1 - \varepsilon$. Formally we have $\sup_{\pi_1 \in \Pi_1^{\mathcal{C}}} \inf_{\pi_2 \in \Pi_2} \Pr_s^{\pi_1,\pi_2}(\Phi) = 1$.

We abbreviate the winning modes by *Almost* and *Limit*, respectively. We call these winning modes the *qualitative* winning modes. Given a game structure G, for $C_1 \in \{P, U, FP, IP\}$ and $C_2 \in \{M, FM, IM\}$ we denote by $\mathit{Almost}_1^G(C_1, C_2, \Phi)$ (resp. $\mathit{Limit}_1^G(C_1, C_2, \Phi)$) the set of almost-sure (resp. limit-sure) winning states for player 1 in G when the strategy set for player 1 is restricted to $\Pi_1^{C_1} \cap \Pi_1^{C_2}$. If the game structure G is clear from the context we omit the superscript G.

2.5 Mu-calculus, complementation, and levels

Consider a mu-calculus expression $\Psi = \mu X . \psi(X)$ over a finite set S, where $\psi : 2^S \mapsto 2^S$ is monotonic. The least fixpoint $\Psi = \mu X . \psi(X)$ is equal to the limit $\lim_{k\to\infty} X_k$, where $X_0 = \emptyset$, and $X_{k+1} = \psi(X_k)$. For every state $s \in \Psi$, we define the *level* $k \geq 0$ of s to be the integer such that $s \notin X_k$ and $s \in X_{k+1}$. The greatest fixpoint $\Psi = \nu X . \psi(X)$ is equal to the limit $\lim_{k\to\infty} X_k$, where $X_0 = S$, and $X_{k+1} = \psi(X_k)$. For every state $s \notin \Psi$, we define the *level* $k \geq 0$ of s to be the integer such that $s \in X_k$ and $s \notin X_{k+1}$. The *height* of a mu-calculus expression $\lambda X . \psi(X)$, where $\lambda \in \{\mu, \nu\}$, is the least integer h such that $X_h = \lim_{k\to\infty} X_k$. An expression of height h can be computed in $h + 1$ iterations. Given a mu-calculus expression $\Psi = \lambda X . \psi(X)$, where $\lambda \in \{\mu, \nu\}$, the complement $\neg\Psi = S \setminus \Psi$ of λ is given by $\overline{\lambda} X . \neg\psi(\neg X)$, where $\overline{\lambda} = \mu$ if $\lambda = \nu$, and $\overline{\lambda} = \nu$ if $\lambda = \mu$. For details of μ-calculus see [Koz83, EJ91].

Distributions and one-step transitions. Given a state $s \in S$, we denote by $\chi_1^s = \mathcal{D}(\Gamma_1(s))$ and $\chi_2^s = \mathcal{D}(\Gamma_2(s))$ the sets of probability distributions over the moves at s available to player 1 and 2, respectively. Moreover, for $s \in S$, $X \subseteq S$, $\xi_1 \in \chi_1^s$, and $\xi_2 \in \chi_2^s$ we denote by

$$P_s^{\xi_1,\xi_2}(X) = \sum_{a\in\Gamma_1(s)} \sum_{b\in\Gamma_2(s)} \sum_{t\in X} \xi_1(a) \cdot \xi_2(b) \cdot \delta(s, a, b)(t)$$

the one-step probability of a transition into X when players 1 and 2 play at s with distributions ξ_1 and ξ_2, respectively. Given a state s and distributions $\xi_1 \in \chi_1^s$ and $\xi_2 \in \chi_2^s$ we denote by $Dest(s, \xi_1, \xi_2) = \{t \in S \mid P_2^{\xi_1,\xi_2}(t) > 0\}$ the set of states that have positive probability of transition from s when the players play ξ_1 and ξ_2 at s. For actions a and b we have $Dest(s, a, b) = \{t \in S \mid \delta(s, a, b)(t) > 0\}$ as the set of possible successors given a and b. For $A \subseteq \Gamma_1(s)$ and $B \subseteq \Gamma_2(s)$ we have $Dest(s, A, B) = \bigcup_{a\in A, b\in B} Dest(s, a, b)$.

Theorem 1 *The following assertions hold:*

1. [CJH03] *For all turn-based stochastic game structures G with a parity objective Φ we have*

$$\mathit{Almost}_1(P, M, \Phi) = \mathit{Almost}_1(IP, IM, \Phi) = \mathit{Limit}_1(P, M, \Phi) = \mathit{Limit}_1(IP, IM, \Phi)$$

2. [dAH00] *Let G_1 and G_2 be two equivalent game structures with a parity objective Φ, then we have*

   1. $\mathit{Almost}_1^{G_1}(IP, IM, \Phi) = \mathit{Almost}_1^{G_2}(IP, IM, \Phi)$; 2. $\mathit{Limit}_1^{G_1}(IP, IM, \Phi) = \mathit{Limit}_1^{G_2}(IP, IM, \Phi)$

3 Pure, Uniform and Finite-precision Strategies

In this section we present our results for pure, uniform and finite-precision strategies. We start with the characterization for pure strategies.

3.1 Pure strategies

The following result shows that for pure strategies, memoryless strategies are as strong as infinite-memory strategies, and the almost-sure and limit-sure sets coincide.

Proposition 1 *Given a concurrent game structure G and a parity objective Φ we have*

$$\mathit{Almost}_1^G(P, M, \Phi) = \mathit{Almost}_1^G(P, FM, \Phi) = \mathit{Almost}_1^G(P, IM, \Phi) =$$
$$\mathit{Limit}_1^G(P, M, \Phi) = \mathit{Limit}_1^G(P, FM, \Phi) = \mathit{Limit}_1^G(P, IM, \Phi).$$

Proof. The result is obtained as follows: we show that $\mathit{Almost}_1^G(P, M, \Phi) = \mathit{Almost}_1^G(P, IM, \Phi) = \mathit{Limit}_1^G(P, IM, \Phi)$ and all the other equalities follow (by inclusion of strategies). The main argument is as follows: given G we obtain a turn-based stochastic game $\widehat{G}$ where player 1 first choses an action, then player 2 chooses an action, and then the game proceeds as in G. Then it is straightforward to establish that the almost-sure (resp. limit-sure) winning set for pure and infinite-memory strategies in G coincides with the almost-sure (resp. limit-sure) winning set for pure and infinite-memory strategies in $\widehat{G}$. Since $\widehat{G}$ is a turn-based stochastic game, by Theorem 1 (part 1), it follows that the almost-sure and limit-sure winning set in $\widehat{G}$ coincide and they are same for memoryless and infinite-memory strategies.

We now present the formal reduction. Let $G = \langle S, A, \Gamma_1, \Gamma_2, \delta \rangle$ and let the parity objective Φ be described by a priority function p. We construct $\widehat{G} = \langle \widehat{S}, \widehat{A}, \widehat{\Gamma}_1, \widehat{\Gamma}_2, \widehat{\delta} \rangle$ with priority function $\widehat{p}$ as follows:

1. $\widehat{S} = S \cup \{(s, a) \mid s \in S, a \in \Gamma_1(s)\}$;
2. $\widehat{A} = A \cup \{\bot\}$ where $\bot \notin A$;
3. for $s \in \widehat{S} \cap S$ we have $\widehat{\Gamma}_1(s) = \Gamma_1(s)$ and $\widehat{\Gamma}_2(s) = \{\bot\}$; and for $(s, a) \in \widehat{S}$ we have $\widehat{\Gamma}_2((s, a)) = \Gamma_2(s)$ and $\widehat{\Gamma}_1((s, a)) = \{\bot\}$; and
4. for $s \in \widehat{S} \cap S$ and $a \in \Gamma_1(s)$ we have $\widehat{\delta}(s, a, \bot)(s, a) = 1$; and for $(s, a) \in \widehat{S}$ and $b \in \Gamma_2(s)$ we have $\widehat{\delta}((s, a), \bot, b) = \delta(s, a, b)$;
5. the function $\widehat{p}$ in $\widehat{G}$ is as follows: for $s \in \widehat{S} \cap S$ we have $\widehat{p}(s) = p(s)$ and for $(s, a) \in \widehat{S}$ we have $\widehat{p}((s, a)) = p(s)$.

It is straightforward to establish by mapping of pure strategies of player 1 in G and $\widehat{G}$ that

$$\begin{array}{rcl}
(a)\ \mathit{Almost}_1^G(P, M, \Phi) & = & \mathit{Almost}_1^{\widehat{G}}(P, M, \widehat{\Phi}) \cap S, \\
(b)\ \mathit{Almost}_1^G(P, IM, \Phi) & = & \mathit{Almost}_1^{\widehat{G}}(P, IM, \widehat{\Phi}) \cap S, \\
(c)\ \mathit{Limit}_1^G(P, M, \Phi) & = & \mathit{Limit}_1^{\widehat{G}}(P, M, \widehat{\Phi}) \cap S, \\
(d)\ \mathit{Limit}_1^G(P, IM, \Phi) & = & \mathit{Limit}_1^{\widehat{G}}(P, IM, \widehat{\Phi}) \cap S;
\end{array}$$

where $\widehat{\Phi} = \text{Parity}(\widehat{p})$. It follows from Theorem 1 (part 1) that

$$\mathit{Almost}_1^{\widehat{G}}(P, M, \widehat{\Phi}) = \mathit{Almost}_1^{\widehat{G}}(P, IM, \widehat{\Phi}) = \mathit{Limit}_1^{\widehat{G}}(P, M, \widehat{\Phi}) = \mathit{Limit}_1^{\widehat{G}}(P, IM, \widehat{\Phi}).$$

Hence the desired result follows. ∎

Algorithm and complexity. The above proposition gives a linear reduction to turn-based stochastic games. Thus the set $\mathit{Almost}_1(P, M, \Phi)$ can be computed using the algorithms for turn-based stochastic parity games (such as [CJH03]). We have the following results.

Theorem 2 *Given a concurrent game structure G, a parity objective Φ, and a state s, whether $s \in \mathit{Almost}_1(P, \mathit{IM}, \Phi) = \mathit{Limit}_1(P, \mathit{IM}, \Phi)$ can be decided in NP $\cap$ coNP.*

3.2 Uniform and Finite-precision

In this subsection we will present the characterization for uniform and finite-precision strategies.

Example 1 It is easy to show that $\mathit{Almost}_1(P, M, \Phi) \subsetneq \mathit{Almost}_1(U, M, \Phi)$ by considering the *matching penny* game. The game has two states s_0 and s_1. The state s_1 is an *absorbing* state (a state with only self-loop as outgoing edge; see state s_1 of Fig 3) and the goal is to reach s_1 (equivalently infinitely often visit s_1). At s_0 the actions available for both players are $\{a, b\}$. If the actions match the next state is s_1, otherwise s_0. By playing a and b uniformly at random at s_0, the state s_1 is reached with probability 1, whereas for any pure strategy the counter-strategy that plays exactly the opposite action in every round ensures s_1 is never reached. ∎

We now show that uniform memoryless strategies are as powerful as finite-precision infinite-memory strategies and the almost-sure and limit-sure sets coincide for finite-precision strategies. We start with two notations.

Uniformization of a strategy. Given a strategy π_1 for player 1, we define a strategy π_1^u that is obtained from π_1 by uniformization as follows: for all $w \in S^+$ and all $a \in \mathrm{Supp}(\pi_1(w))$ we have $\pi_1^u(w)(a) = \frac{1}{|\mathrm{Supp}(\pi_1(w))|}$. We will use the following notation for uniformization: $\pi_1^u = \mathsf{unif}(\pi_1)$.

b-finite-precision strategies. Given $b \in \mathbb{N}$, a strategy is b-finite-precision if for all $x \in S^+$ and all actions a we have $\pi(x)(a) = \frac{i}{j}$, where $i, j \in \mathbb{N}$ and $0 \le i \le j \le b$ and $j > 0$.

Proposition 2 *Given a concurrent game structure G and a parity objective Φ we have*

$$\begin{aligned}
&\mathit{Almost}_1^G(U, M, \Phi) = \mathit{Almost}_1^G(U, \mathit{FM}, \Phi) = \mathit{Almost}_1^G(U, \mathit{IM}, \Phi) = \\
&\mathit{Limit}_1^G(U, M, \Phi) = \mathit{Limit}_1^G(U, \mathit{FM}, \Phi) = \mathit{Limit}_1^G(U, \mathit{IM}, \Phi) = \\
&\mathit{Almost}_1^G(\mathit{FP}, M, \Phi) = \mathit{Almost}_1^G(\mathit{FP}, \mathit{FM}, \Phi) = \mathit{Almost}_1^G(\mathit{FP}, \mathit{IM}, \Phi) = \\
&\mathit{Limit}_1^G(\mathit{FP}, M, \Phi) = \mathit{Limit}_1^G(\mathit{FP}, \mathit{FM}, \Phi) = \mathit{Limit}_1^G(\mathit{FP}, \mathit{IM}, \Phi)
\end{aligned}$$

Proof. The result is obtained as follows: we show that $\mathit{Almost}_1^G(U, M, \Phi) = \mathit{Almost}_1^G(\mathit{FP}, \mathit{IM}, \Phi) = \mathit{Limit}_1^G(\mathit{FP}, \mathit{IM}, \Phi)$ and all the other equalities follow (by inclusion of strategies). The key argument is as follows: fix a bound b, and we consider the set of b-finite-precision strategies in G. Given G we obtain a turn-based stochastic game $\widetilde{G}$ where player 1 first chooses a b-finite-precision distribution, then player 2 chooses an action, and then the game proceeds as in G. Then we establish that the almost-sure (resp. limit-sure) winning set for b-finite-precision and infinite-memory strategies in G coincides with the almost-sure (resp. limit-sure) winning set for b-finite-precision and infinite-memory strategies in $\widetilde{G}$. Since $\widetilde{G}$ is a turn-based stochastic game, by Theorem 1, it follows that the almost-sure and limit-sure winning set in $\widetilde{G}$ coincide and they are same for memoryless and infinite-memory strategies. Thus we obtain a b-finite-precision memoryless almost-sure winning strategy π_1 in G and then we show the uniform memoryless

$\pi_1^u = \mathsf{unif}(\pi_1)$ obtained from uniformization of π_1^u is a uniform memoryless almost-sure winning strategy in G. Thus it follows that for any finite-precision infinite-memory almost-sure winning strategy, there is a uniform memoryless almost-sure winning strategy.

We now present the formal reduction. Let $G = \langle S, A, \Gamma_1, \Gamma_2, \delta\rangle$ and let the parity objective Φ be described by a priority function p. For a given bound b, let $\widetilde{f}(s,b) = \{f : \Gamma_1(s) \mapsto [0,1] \mid \forall a \in \Gamma_1(s)$ we have $f(a) = \frac{i}{j}, i,j \in \mathbb{N}, 0 \leq i \leq j \leq b, j > 0$ and $\sum_{a\in\Gamma_1(s)} f(a) = 1\}$ denote the set of b-finite-precision distributions at s. We construct $\widetilde{G} = \langle \widetilde{S}, \widetilde{A}, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\delta}\rangle$ with priority function $\widetilde{p}$ as follows:

1. $\widetilde{S} = S \cup \{(s,f) \mid s \in S, f \in \widetilde{f}(s,b)\}$;
2. $\widetilde{A} = A \cup \{f \mid s \in S, f \in \widetilde{f}(s,b)\} \cup \{\bot\}$ where $\bot \notin A$;
3. for $s \in \widetilde{S} \cap S$ we have $\widetilde{\Gamma}_1(s) = \widetilde{f}(s,b)$ and $\widetilde{\Gamma}_2(s) = \{\bot\}$; and for $(s,f) \in \widetilde{S}$ we have $\widetilde{\Gamma}_2((s,f)) = \Gamma_2(s)$ and $\widetilde{\Gamma}_1((s,f)) = \{\bot\}$; and
4. for $s \in \widetilde{S} \cap S$ and $f \in \widetilde{f}(s,b)$ we have $\widetilde{\delta}(s,f,\bot)(s,f) = 1$; and for $(s,f) \in \widetilde{S}$, $b \in \Gamma_2(s)$ and $t \in S$ we have $\widehat{\delta}((s,f),\bot,b)(t) = \sum_{a\in\Gamma_1(s)} f(a)\cdot\delta(s,a,b)(t)$;
5. the function $\widetilde{p}$ in $\widetilde{G}$ is as follows: for $s \in \widetilde{S} \cap S$ we have $\widetilde{p}(s) = p(s)$ and for $(s,f) \in \widetilde{S}$ we have $\widetilde{p}((s,f)) = p(s)$.

Observe that given $b \in \mathbb{N}$ the set $\widetilde{f}(s,b)$ is finite and thus $\widetilde{G}$ is a finite-state turn-based stochastic game. It is straightforward to establish mapping of b-finite-precision strategies of player 1 in G and with pure strategies in $\widehat{G}$, i.e., we have

$$\begin{aligned}
(a)\ \mathit{Almost}_1^G(bFP, M, \Phi) &= \mathit{Almost}_1^{\widetilde{G}}(P, M, \widetilde{\Phi}) \cap S,\\
(b)\ \mathit{Almost}_1^G(bFP, IM, \Phi) &= \mathit{Almost}_1^{\widetilde{G}}(P, IM, \widetilde{\Phi}) \cap S,\\
(c)\ \mathit{Limit}_1^G(bFP, M, \Phi) &= \mathit{Limit}_1^{\widetilde{G}}(P, M, \widetilde{\Phi}) \cap S,\\
(d)\ \mathit{Limit}_1^G(bFP, IM, \Phi) &= \mathit{Limit}_1^{\widetilde{G}}(P, IM, \widetilde{\Phi}) \cap S,
\end{aligned}$$

where $\widetilde{\Phi} = \text{Parity}(\widetilde{p})$ and bFP denote the set of b-finite-precision strategies in G. By Theorem 1 we have

$$\mathit{Almost}_1^{\widetilde{G}}(P, M, \widetilde{\Phi}) = \mathit{Almost}_1^{\widetilde{G}}(P, IM, \widetilde{\Phi}) = \mathit{Limit}_1^{\widetilde{G}}(P, M, \widetilde{\Phi}) = \mathit{Limit}_1^{\widetilde{G}}(P, IM, \widetilde{\Phi}).$$

Consider a pure memoryless strategy $\widetilde{\pi}_1$ in $\widetilde{G}$ that is almost-sure winning from $Q = \mathit{Almost}_1^{\widetilde{G}}(P, M, \widetilde{\Phi})$, and let π_1 be the corresponding b-finite-precision memoryless strategy in G. Consider the uniform memoryless strategy $\pi_1^u = \mathsf{unif}(\pi_1)$ in G. The strategy π_1 is an almost-sure winning strategy from $Q \cap S$. The player-2 MDP G_{π_1} and $G_{\pi_1^u}$ are equivalent, i.e., $G_{\pi_1} \equiv G_{\pi_1^u}$ and hence it follows from Theorem 1 that π_1^u is an almost-sure winning strategy for all states in $Q \cap S$. Hence the desired result follows. ∎

Computation of *Almost*$_1(U, M, \Phi)$. It follows from Proposition 2 that the computation of *Almost*$_1(U, M, \Phi)$ can be achieved by a reduction to turn-based stochastic game. We now present the main technical result of this subsection which presents a symbolic algorithm to compute *Almost*$_1(U, M, \Phi)$. The symbolic algorithm developed in this section is crucial for analysis of infinite-precision finite-memory strategies, where the reduction to turn-based stochastic game cannot be applied. The symbolic algorithm is obtained via μ-calculus formula characterization. We first discuss the comparison of our proof with the results of [CdAH11] and then discuss why the recursive characterization of turn-based games fails in concurrent games.

Comparison with [CdAH11]. Our proof structure based on induction on the structure of μ-calculus formulas is similar to the proofs in [CdAH11]. In some aspects the proofs are tedious adaptation but in most cases there are many subtle issues and we point them below. First, in our proof the predecessor operators are different from the predecessor operators of [CdAH11]. Second, in our proof from the μ-calculus formulas we construct uniform memoryless strategies as compared to infinite memory strategies in [CdAH11]. Finally, since our predecessor operators are different the proof for complementation of the predecessor operators (which is a crucial component of the proof) is completely different.

Failue of recursive characterization. In case of turn-based games there are recursive characterization of the winning set with attractors (or alternating reachability). However such characterization fails in case of concurrent games. The intuitive reason is as follows: once an attractor is taken it may rule out certain action pairs (for example, action pair a_1 and b_1 must be ruled out, whereas action pair a_1 and b_2 may be allowed in the remaining game graph), and hence the complement of an attractor maynot satisfy the required sub-game property. For details, see examples in [dAH00, dAHK07] why the recursive characterization fails.

Strategy constructions. Since the recursive characterization of turn-based games fails for concurrent games, our results show that the generalization of the μ-calculus formulas for turn-based games can characterize the desired winning sets. Moreover, our correctness proofs that establish the correctness of the μ-calculus formulas present explicit witness strategies from the μ-calculus formulas. Morover, in all cases the witness counter strategies for player 2 is memoryless, and thus our results answer questions related to bounded rationality for both players.

We now introduce the predecessor operators for the μ-calculus formula required for our symbolic algorithms.

Basic predecessor operators. We recall the *predecessor* operators Pre_1 (pre) and Apre_1 (almost-pre), defined for all $s \in S$ and $X, Y \subseteq S$ by:

$$\begin{array}{rcl} \text{Pre}_1(X) & = & \{s \in S \mid \exists \xi_1 \in \chi_1^s \,.\, \forall \xi_2 \in \chi_2^s \,.\, P_s^{\xi_1,\xi_2}(X) = 1\}; \\ \text{Apre}_1(Y, X) & = & \{s \in S \mid \exists \xi_1 \in \chi_1^s \,.\, \forall \xi_2 \in \chi_1^s \,.\, P_s^{\xi_1,\xi_2}(Y) = 1 \wedge P_s^{\xi_1,\xi_2}(X) > 0\} \,. \end{array}$$

Intuitively, the $\text{Pre}_1(X)$ is the set of states such that player 1 can ensure that the next state is in X with probability 1, and $\text{Apre}_1(Y, X)$ is the set of states such that player 1 can ensure that the next state is in Y with probability 1 and in X with positive probability.

Principle of general predecessor operators. While the operators Apre and Pre suffice for solving Büchi games, for solving general parity games, we require predecessor operators that are best understood as the combination of the basic predecessor operators. We use the operators $\uplus\!\!\!\!*$ and $\Cap\!\!\!\!*$ to combine predecessor operators; the operators $\uplus\!\!\!\!*$ and $\Cap\!\!\!\!*$ are different from the usual union $\cup$ and intersection $\cap$. Roughly, let α and β be two set of states for two predecessor operators, then the set $\alpha \Cap\!\!\!\!* \beta$ requires that the distributions of player 1 satisfy the conjunction of the conditions stipulated by α and β; similarly, $\uplus\!\!\!\!*$ corresponds to disjunction. We first introduce the operator $\text{Apre} \uplus\!\!\!\!* \text{Pre}$. For all $s \in S$ and $X_1, Y_0, Y_1 \subseteq S$, we define

$$\text{Apre}_1(Y_1, X_1) \uplus\!\!\!\!* \text{Pre}_1(Y_0) = \left\{ s \in S \mid \exists \xi_1 \in \chi_1^s . \forall \xi_2 \in \chi_2^s . \left[\begin{array}{c} (P_s^{\xi_1,\xi_2}(X_1) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_1) = 1) \\ \bigvee \\ P_s^{\xi_1,\xi_2}(Y_0) = 1 \end{array} \right] \right\} .$$

Note that the above formula corresponds to a disjunction of the predicates for Apre_1 and Pre_1. However, it is important to note that the distributions ξ_1 for player 1 to satisfy (ξ_2 for player 2 to falsify) the predicate must be *the same*. In other words, $\text{Apre}_1(Y_1, X_1) \uplus\!\!\!\!* \text{Pre}_1(Y_0)$ is *not* equivalent to $\text{Apre}_1(Y_1, X_1) \cup \text{Pre}_1(Y_0)$.

General predecessor operators. We first introduce two predecessor operators as follows:

$$\begin{aligned}&\mathrm{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})\\&= \mathrm{Apre}_1(Y_n, X_n) \uplus \mathrm{Apre}_1(Y_{n-1}, X_{n-1}) \uplus \cdots \uplus \mathrm{Apre}_1(Y_{n-i}, X_{n-i});\end{aligned}$$

$$\begin{aligned}&\mathrm{APreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})\\&= \mathrm{Apre}_1(Y_n, X_n) \uplus \mathrm{Apre}_1(Y_{n-1}, X_{n-1}) \uplus \cdots \uplus \mathrm{Apre}_1(Y_{n-i}, X_{n-i}) \uplus \mathrm{Pre}_1(Y_{n-i-1}).\end{aligned}$$

The formal expanded definitions of the above operators are as follows:

$$\mathrm{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) =$$
$$\left\{ s \in S \mid \exists \xi_1 \in \chi_1^s . \forall \xi_2 \in \chi_2^s . \begin{bmatrix} (P_s^{\xi_1,\xi_2}(X_n) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-i}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-i}) = 1) \end{bmatrix} \right\}.$$

$$\mathrm{APreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) =$$
$$\left\{ s \in S \mid \exists \xi_1 \in \chi_1^s . \forall \xi_2 \in \chi_2^s . \begin{bmatrix} (P_s^{\xi_1,\xi_2}(X_n) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-i}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-i}) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-i-1}) = 1) \end{bmatrix} \right\}.$$

Observe that the above definition can be inductively written as follows:

1. We have $\mathrm{APreOdd}_1(0, Y_n, X_n) = \mathrm{Apre}_1(Y_n, X_n)$ and for $i \geq 1$ we have

$$\begin{aligned}&\mathrm{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})\\&= \mathrm{Apre}_1(Y_n, X_n) \uplus \mathrm{APreOdd}_1(i-1, Y_{n-1}, X_{n-1}, \ldots, Y_{n-i}, X_{n-i})\end{aligned}$$

2. We have $\mathrm{APreEven}_1(0, Y_n, X_n, Y_{n-1}) = \mathrm{Apre}_1(Y_n, X_n) \uplus \mathrm{Pre}_1(Y_{n-1})$ and for $i \geq 1$ we have

$$\begin{aligned}&\mathrm{APreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})\\&= \mathrm{Apre}_1(Y_n, X_n) \uplus \mathrm{APreEven}_1(i-1, Y_{n-1}, X_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})\end{aligned}$$

Dual operators. The *predecessor* operators Pospre_2 (positive-pre) and Apre_2 (almost-pre), defined for all $s \in S$ and $X, Y \subseteq S$ by:

$$\begin{aligned}
\text{Pospre}_2(X) &= \{s \in S \mid \forall \xi_1 \in \chi_1^s \,.\, \exists \xi_2 \in \chi_2^s \,.\, P_s^{\xi_1,\xi_2}(X) > 0\}; \\
\text{Apre}_2(Y, X) &= \{s \in S \mid \forall \xi_1 \in \chi_1^s \,.\, \exists \xi_2 \in \chi_1^s \,.\, P_s^{\xi_1,\xi_2}(Y) = 1 \wedge P_s^{\xi_1,\xi_2}(X) > 0\} \,.
\end{aligned}$$

Observe that player 2 is only required to play counter-distributions ξ_2 against player 1 distributions ξ_1. We now introduce two positive predecessor operators as follows:

$$\begin{aligned}
&\text{PosPreOdd}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) \\
&= \text{Pospre}_2(Y_n) \uplus_* \text{Apre}_2(X_n, Y_{n-1}) \uplus_* \cdots \uplus_* \text{Apre}_2(X_{n-i+1}, Y_{n-i}) \uplus_* \text{Pre}_2(X_{n-i})
\end{aligned}$$

$$\begin{aligned}
&\text{PosPreEven}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) \\
&= \text{Pospre}_2(Y_n) \uplus_* \text{Apre}_2(X_n, Y_{n-1}) \\
&\qquad \uplus_* \cdots \uplus_* \text{Apre}_2(X_{n-i+1}, Y_{n-i}) \uplus_* \text{Apre}_2(X_{n-i}, Y_{n-i-1})
\end{aligned}$$

The formal expanded definitions of the above operators are as follows:

$$\text{PosPreOdd}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) =$$

$$\left\{ s \in S \mid \forall \xi_1 \in \chi_1^s . \exists \xi_2 \in \chi_2^s . \begin{bmatrix} (P_s^{\xi_1,\xi_2}(Y_n) > 0) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-2}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-i}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_{n-i+1}) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-i}) = 1) \end{bmatrix} \right\} .$$

$$\text{PosPreEven}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) =$$

$$\left\{ s \in S \mid \forall \xi_1 \in \chi_1^s . \exists \xi_2 \in \chi_2^s . \begin{bmatrix} (P_s^{\xi_1,\xi_2}(Y_n) > 0) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-2}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-i-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_{n-i}) = 1) \end{bmatrix} \right\} .$$

The above definitions can be alternatively written as follows

$$\begin{aligned}
\text{PosPreOdd}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) &= \\
\text{Pospre}_2(Y_n) \quad \uplus_* \quad & \text{APreEven}_2(i-1, X_n, Y_{n-1}, \ldots, X_{n-i+1}, Y_{n-i}, X_{n-i});
\end{aligned}$$

$$\text{PosPreEven}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) = \text{Pospre}_2(Y_n) \circledast \text{APreOdd}_2(i, X_n, Y_{n-1}, \ldots, X_{n-i}, Y_{n-i-1}).$$

Remark 1 *Observe that if the predicate* $\text{Pospre}_2(Y_n)$ *is removed from the predecessor operator* *PosPreOdd*$_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$ *(resp.* *PosPreEven*$_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$*), then we obtain the operator* *APreEven*$_2(i-1, X_n, Y_{n-1}, \ldots, X_{n-i+1}, Y_{n-i}, X_{n-i})$ *(resp.* *APreOdd*$_2(i, X_n, Y_{n-1}, \ldots, X_{n-i}, Y_{n-i-1})$*).*

We first show how to characterize the set of almost-sure winning states for uniform memoryless strategies and its complement for parity games using the above predecessor operators. We will prove the following result by induction.

1. *Case 1.* For a parity function $p : S \mapsto [0..2n-1]$ the following assertions hold.

   (a) For all $T \subseteq S$ we have $W \subseteq \textit{Almost}_1(U, M, \text{Parity}(p) \cup \Diamond T)$, where W is defined as follows:

$$\nu Y_n.\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0. \begin{bmatrix} T \\ \cup \\ B_{2n-1} \cap \text{APreOdd}_1(0, Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \text{APreEven}_1(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \text{APreOdd}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \text{APreEven}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \text{APreOdd}_1(n-1, Y_n, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \text{APreEven}_1(n-1, Y_n, X_n, \ldots, Y_1, X_1, Y_0) \end{bmatrix}$$

   We refer to the above expression as the *almost-expression* for case 1. If in the above formula we replace APreOdd_1 by APreOdd_2 and APreEven_1 by APreEven_2 then we obtain the *dual almost-expression* for case 1. From the same argument as correctness of the almost-expression and the fact that counter-strategies for player 2 are against memoryless strategies for player 1 we obtain that if the dual almost-expression is W_D for $T = \emptyset$, then $W_D \subseteq \{s \in S \mid \forall \pi_1 \in \Pi_1^M.\exists \pi_2 \in \Pi_2.\ \Pr_s^{\pi_1,\pi_2}(\text{coParity}(p)) = 1\}$.

   (b) We have $Z \subseteq \neg\textit{Almost}_1(U, M, \text{Parity}(p))$, where Z is defined as follows

$$\mu Y_n.\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_1.\nu X_1.\mu Y_0. \begin{bmatrix} B_{2n-1} \cap \text{PosPreOdd}_2(0, Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \text{PosPreEven}_2(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \text{PosPreOdd}_2(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \text{PosPreEven}_2(1, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \text{PosPreOdd}_2(n-1, Y_n, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \text{PosPreEven}_2(n-1, Y_n, X_n, \ldots, Y_1, X_1, Y_0) \end{bmatrix}$$

We refer to the above expression as the *positive-expression* for case 1.

2. *Case 2.* For a parity function $p : S \mapsto [1..2n]$ the following assertions hold.

   (a) For all $T \subseteq S$ we have $W \subseteq \mathit{Almost}_1(U, M, \text{Parity}(p) \cup \Diamond T)$, where W is defined as follows:

$$\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0.\mu X_0 \begin{bmatrix} T \\ \cup \\ B_{2n} \cap \text{Pre}_1(Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{APreEven}_1(0, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \text{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{bmatrix}$$

   We refer to the above expression as the almost-expression for case 2. If in the above formula we replace APreOdd_1 by APreOdd_2 and APreEven_1 by APreEven_2 then we obtain the dual almost-expression for case 2. Again, if the dual almost-expression is W_D for $T = \emptyset$, then $W_D \subseteq \{s \in S \mid \forall \pi_1 \in \Pi_1^M.\exists \pi_2 \in \Pi_2.\ \text{Pr}_s^{\pi_1,\pi_2}(\text{coParity}(p)) = 1\}$.

   (b) We have $Z \subseteq \neg \mathit{Almost}_1(U, M, \text{Parity}(p))$, where Z is defined as follows

$$\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_1.\nu X_1.\mu Y_0.\nu X_0 \begin{bmatrix} B_{2n} \cap \text{Pospre}_2(Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{PosPreOdd}_2(0, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{PosPreEven}_2(0, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{PosPreOdd}_2(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \text{PosPreEven}_2(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{PosPreOdd}_2(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{bmatrix}$$

   We refer to the above expression as the positive-expression for case 2.

The comparison to Emerson-Jutla μ-calculus formula for turn-based games. We compare our μ-calculus formula with the μ-calculus formula of Emerson-Jutla [EJ91] to give an intuitive idea of the construction of the formula. We first present the formula for Case 2 and then for Case 1.

Case 2. For turn-based deterministic games with parity function $p : S \to [1..2n]$, it follows from the results of Emerson-Jutla [EJ91], that the sure-winning (that is equivalent to the almost-sure winning) set for the objective Parity$(p) \cup \Diamond T$ is given by the following μ-calculus formula:

$$\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0.\mu X_0 \begin{bmatrix} T \\ \cup \\ B_{2n} \cap \text{Pre}_1(Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{Pre}_1(X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{Pre}_1(Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{Pre}_1(X_{n-2}) \\ \vdots \\ B_2 \cap \text{Pre}_1(Y_0) \\ \cup \\ B_1 \cap \text{Pre}_1(X_0) \end{bmatrix}$$

The formula for the almost-expression for case 2 is similar to the above μ-calculus formula and is obtained by replacing the Pre_1 operators with appropriate APreOdd_1 and APreEven_1 operators.

Case 1. For turn-based deterministic games with parity function $p : S \to [0..2n-1]$, it follows from the results of Emerson-Jutla [EJ91], that the sure-winning (that is equivalent to the almost-sure winning) set for the objective Parity$(p) \cup \Diamond T$ is given by the following μ-calculus formula:

$$\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0. \begin{bmatrix} T \\ \cup \\ B_{2n-1} \cap \text{Pre}_1(X_n) \\ \cup \\ B_{2n-2} \cap \text{Pre}_1(Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \text{Pre}_1(X_{n-1}) \\ \cup \\ B_{2n-4} \cap \text{Pre}_1(Y_{n-2}) \\ \vdots \\ B_1 \cap \text{Pre}_1(X_1) \\ \cup \\ B_0 \cap \text{Pre}_1(Y_0) \end{bmatrix}$$

The formula for the almost-expression for case 1 is similar to the above μ-calculus formula and is obtained by (a) adding one quantifier alternation νY_n; and (b) replacing the Pre_1 operators with appropriate APreOdd_1 and APreEven_1 operators.

Proof structure. The base case follows from the coBüchi and Büchi case: it follows from the results of [dAH00] since for Büchi and coBüchi objectives, uniform memoryless almost-sure winning strategies exist and our μ-calculus formula coincide with the μ-calculus formula to describe the almost-sure winning set for Büchi and coBüchi objectives. The proof of induction proceeds in four steps as follows:

1. *Step 1.* We assume the correctness of case 1 and case 2, and then extend the result to parity objective with parity function $p : S \mapsto [0..2n]$, i.e., we add a max even priority. The result is obtained as follows:

for the correctness of the almost-expression we use the correctness of case 1 and for complementation we use the correctness of case 2.

2. *Step 2.* We assume the correctness of step 1 and extend the result to parity objectives with parity function $p : S \mapsto [1..2n+1]$, i.e., we add a max odd priority. The result is obtained as follows: for the correctness of the almost-expression we use the correctness of case 2 and for complementation we use the correctness of step 1.

3. *Step 3.* We assume correctness of step 2 and extend the result to parity objectives with parity function $p : S \mapsto [1..2n+2]$. This step adds a max even priority and the proof will be similar to step 1. The result is obtained as follows: for the correctness of the almost-expression we use the correctness of step 2 and for complementation we use the correctness of step 1.

4. *Step 4.* We assume correctness of step 3 and extend the result to parity objectives with parity function $p : S \mapsto [0..2n+1]$. This step adds a max odd priority and the proof will be similar to step 2. The result is obtained as follows: for the correctness of the almost-expression we use the correctness of step 1 and for complementation we use the correctness of step 3.

We first present two technical lemmas that will be used in the correctness proofs. First we define prefix-independent events.

Prefix-independent events. We say that an event or objective is *prefix-independent* if it is independent of all finite prefixes. Formally, an event or objective $\mathcal{A}$ is prefix-independent if, for all $u, v \in S^*$ and $\omega \in S^\omega$, we have $u\omega \in \mathcal{A}$ iff $v\omega \in \mathcal{A}$. Observe that parity objectives are defined based on the states that appear infinitely often along a play, and hence independent of all finite prefixes, so that, parity objectives are prefix-independent objectives.

Lemma 1 (Basic Apre principle). *Let $X \subseteq Y \subseteq Z \subseteq S$ and $s \in S$ be such that $Y = X \cup \{s\}$ and $s \in Apre_1(Z, X)$. For all prefix-independent events $\mathcal{A} \subseteq \Box(Z \setminus Y)$, the following assertion holds:*

> *Assume that there exists a uniform memoryless $\pi_1 \in \Pi_1^U \cap \Pi_1^M$ such that for all $\pi_2 \in \Pi_2$ and for all $z \in Z \setminus Y$ we have*
> $$\Pr_z^{\pi_1,\pi_2}(\mathcal{A} \cup \Diamond Y) = 1.$$
> *Then there exists a uniform memoryless $\pi_1 \in \Pi_1^U \cap \Pi_1^M$ such that for all $\pi_2 \in \Pi_2$ we have*
> $$\Pr_s^{\pi_1,\pi_2}(\mathcal{A} \cup \Diamond X) = 1.$$

Proof. Since $s \in \text{Apre}_1(Z, X)$, player 1 can play a uniform memoryless distribution ξ_1 at s to ensure that the probability of staying in Z is 1 and with positive probability $\eta > 0$ the set X is reached. In $Z \setminus Y$ player 1 fixes a uniform memoryless strategy to ensure that $\mathcal{A} \cup \Diamond Y$ is satisfied with probability 1. Fix a counter strategy π_2 for player 2. If s is visited infinitely often, then since there is a probability of at least $\eta > 0$ to reach X, it follows that X is reached with probability 1. If s is visited finitely often, then from some point on $\Box(Z \setminus Y)$ is satisfied, and then $\mathcal{A}$ is ensured with probability 1. Thus the desired result follows. ∎

Lemma 2 (Basic principle of repeated reachability). *Let $T \subseteq S$, $B \subseteq S$ and $W \subseteq S$ be sets and $\mathcal{A}$ be a prefix-independent objective such that*

$$W \subseteq Almost_1(U, M, \Diamond T \cup \Diamond(B \cap Pre_1(W)) \cup \mathcal{A}).$$

Then

$$W \subseteq \mathit{Almost}_1(U, M, \diamond T \cup \Box\diamond B \cup \mathcal{A}).$$

Proof. Let $Z = B \cap \text{Pre}_1(W)$. For all states $s \in W \setminus (Z \cup T)$, there is a uniform memoryless player 1 strategy π_1 that ensures that against all player 2 strategies π_2 we have

$$\text{Pr}_s^{\pi_1,\pi_2}\big(\diamond(T \cup Z) \cup \mathcal{A}\big) = 1.$$

For all states in Z player 1 can ensure that the successor state is in W (since $\text{Pre}_1(W)$ holds in Z). Consider a strategy π_1^* as follows: for states $s \in Z$ play a uniform memoryless strategy for $\text{Pre}_1(W)$ to ensure that the next state is in W; for states $s \in W \setminus (Z \cup T)$ play the uniform memoryless strategy π_1. Let us denote by $\diamond_k Z \cup \diamond T$ to be the set of paths that visits Z at least k-times or visits T at least once. Observe that $\lim_{k\to\infty}\big(\diamond_k Z \cup \diamond T\big) \subseteq \Box\diamond B \cup \diamond T$. Hence for all $s \in W$ and for all $\pi_2 \in \Pi_2$ we have

$$\begin{aligned} \text{Pr}_s^{\pi_1^*,\pi_2}(\Box\diamond B \cup \diamond T \cup \mathcal{A}) &\geq \text{Pr}_s^{\pi_1^*,\pi_2}\big(\diamond Z \cup \diamond T \cup \mathcal{A}\big) \cdot \prod_{k=1}^{\infty} \text{Pr}_s^{\pi_1^*,\pi_2}\big(\diamond_{k+1} Z \cup \diamond T \cup \mathcal{A} \mid \diamond_k Z \cup \diamond T \cup \mathcal{A}\big) \\ &= 1. \end{aligned}$$

The desired result follows. ∎

Correctness of step 1. We now proceed with the proof of step 1 and by inductive hypothesis we will assume that case 1 and case 2 hold.

Lemma 3 *For a parity function $p : S \mapsto [0..2n]$, and for all $T \subseteq S$, we have $W \subseteq$ $\mathit{Almost}_1(U, M, \mathit{Parity}(p) \cup \diamond T)$, where W is defined as follows:*

$$\nu Y_n.\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0. \begin{bmatrix} T \\ \cup \\ B_{2n} \cap \mathit{Pre}_1(Y_n) \\ \cup \\ B_{2n-1} \cap \mathit{APreOdd}_1(0, Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \mathit{APreEven}_1(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \mathit{APreOdd}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \mathit{APreEven}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \mathit{APreOdd}_1(n-1, Y_n, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \mathit{APreEven}_1(n-1, Y_n, X_n, \ldots, Y_1, X_1, Y_0) \end{bmatrix}$$

Proof. We first present the intuitive explanation of obtaining the μ-calculus formula.

Intuitive explanation of the μ-calculus formula. The μ-calculus formula of the lemma is obtained from the almost-expression for case 1 by just adding the expression $B_{2n} \cap \text{Pre}_1(Y_n)$.

To prove the result we first rewrite W as follows:

$$\nu Y_n.\mu X_n.\nu Y_{n-1}\mu X_{n-1}\cdots \nu Y_1.\mu X_1.\nu Y_0. \left[\begin{array}{c} T\cup (B_{2n}\cap \text{Pre}_1(W)) \\ \cup \\ B_{2n-1}\cap \text{APreOdd}_1(0,Y_n,X_n) \\ \cup \\ B_{2n-2}\cap \text{APreEven}_1(0,Y_n,X_n,Y_{n-1}) \\ \cup \\ B_{2n-3}\cap \text{APreOdd}_1(1,Y_n,X_n,Y_{n-1},X_{n-1}) \\ \cup \\ B_{2n-4}\cap \text{APreEven}_1(1,Y_n,X_n,Y_{n-1},X_{n-1},Y_{n-2}) \\ \vdots \\ B_1\cap \text{APreOdd}_1(n-1,Y_n,X_n,\ldots,Y_1,X_1) \\ \cup \\ B_0\cap \text{APreEven}_1(n-1,Y_n,X_n,\ldots,Y_1,X_1,Y_0) \end{array}\right]$$

The rewriting is obtained as follows: since W is the fixpoint Y_n, we replace Y_n in the $B_{2n}\cap \text{Pre}_1(Y_n)$ by W. Treating $T\cup (B_{2n}\cap \text{Pre}_1(W))$, as the set T for the almost-expression for case 1, we obtain from the inductive hypothesis that

$$W\subseteq \textit{Almost}_1(U,M,\text{Parity}(p)\cup \Diamond(T\cup (B_{2n}\cap \text{Pre}_1(W)))).$$

By Lemma 2, with $B=B_{2n}$ and $\mathcal{A}=\text{Parity}(p)$ we obtain that

$$W\subseteq \textit{Almost}_1(U,M,\text{Parity}(p)\cup \Diamond T\cup \Box\Diamond B_{2n}).$$

Since B_{2n} is the maximal priority and it is even we have $\Box\Diamond B_{2n} \subseteq \text{Parity}(p)$. Hence $W \subseteq$ $\textit{Almost}_1(U,M,\text{Parity}(p)\cup \Diamond T)$ and the result follows. ∎

Lemma 4 *For a parity function $p: S\mapsto [0..2n]$, we have $Z\subseteq \neg \textit{Almost}_1(U,M,\textit{Parity}(p))$, where Z is defined as follows*

$$\mu Y_n.\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_1.\nu X_1.\mu Y_0. \left[\begin{array}{c} B_{2n}\cap \textit{Pospre}_2(Y_n) \\ \cup \\ B_{2n-1}\cap \textit{PosPreOdd}_2(0,Y_n,X_n) \\ \cup \\ B_{2n-2}\cap \textit{PosPreEven}_2(0,Y_n,X_n,Y_{n-1}) \\ \cup \\ B_{2n-3}\cap \textit{PosPreOdd}_2(1,Y_n,X_n,Y_{n-1},X_{n-1}) \\ \cup \\ B_{2n-4}\cap \textit{PosPreEven}_2(1,Y_n,X_n,Y_{n-1},X_{n-1},Y_{n-2}) \\ \vdots \\ B_1\cap \textit{PosPreOdd}_2(n-1,Y_n,X_n,\ldots,Y_1,X_1) \\ \cup \\ B_0\cap \textit{PosPreEven}_2(n-1,Y_n,X_n,\ldots,Y_1,X_1,Y_0) \end{array}\right]$$

Proof. For $k \geq 0$, let Z_k be the set of states of level k in the above μ-calculus expression. We will show that in Z_k for every memoryless strategy for player 1, player 2 can ensure that either Z_{k-1} is reached with positive probability or else coParity(p) is satisfied with probability 1. Since $Z_0 = \emptyset$, it would follow by induction that $Z_k \cap \textit{Almost}_1(U, M, \text{Parity}(p)) = \emptyset$ and the desired result will follow.

We simplify the computation of Z_k given Z_{k-1} and allow that Z_k is obtained from Z_{k-1} in the following two ways.

1. Add a set states satisfying $B_{2n} \cap \text{Pospre}_2(Z_{k-1})$, and if such a non-emptyset is added, then clearly against any memoryless stratgy for player 1, player 2 can ensure from Z_k that Z_{k-1} is reached with positive probability. Thus the inductive case follows.

2. Add a set of states satisfying the following condition:

$$\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_1.\nu X_1.\mu Y_0. \left[\begin{array}{c} B_{2n-1} \cap \text{PosPreOdd}_2(0, Z_{k-1}, X_n) \\ \cup \\ B_{2n-2} \cap \text{PosPreEven}_2(0, Z_{k-1}, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \text{PosPreOdd}_2(1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \text{PosPreEven}_2(1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \text{PosPreOdd}_2(n-1, Z_{k-1}, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \text{PosPreEven}_2(n-1, Z_{k-1}, X_n, \ldots, Y_1, X_1, Y_0) \end{array}\right]$$

If the probability of reaching to Z_{k-1} is not positive, then the following conditions hold:

- If the probability to reach Z_{k-1} is not positive, then the predicate $\text{Pospre}_2(Z_{k-1})$ vanishes from the predecessor operator $\text{PosPreOdd}_2(i, Z_{k-1}, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i})$, and thus the operator simplifies to the simpler predecessor operator $\text{APreEven}_2(i-1, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i})$.
- If the probability to reach Z_{k-1} is not positive, then the $\text{Pospre}_2(Z_{k-1})$ vanishes from the predecessor operator $\text{PosPreEven}_2(i, Z_{k-1}, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$, and thus the operator simplifies to the predecessor operator $\text{APreOdd}_2(i, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$.

Hence either the probability to reach Z_{k-1} is positive, or if the probability to reach Z_{k-1} is not positive, then the above μ-calculus expression simplifies to

$$Z^* = \nu X_n.\mu Y_{m-1}\nu X_{m-1}\cdots \mu Y_1.\nu X_1.\mu Y_0. \left[\begin{array}{c} B_{2n-1} \cap \text{Pre}_2(X_n) \\ \cup \\ B_{2n-2} \cap \text{APreOdd}_2(0, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \text{APreEven}_2(1, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \text{APreOdd}_2(1, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \text{APreEven}_2(n-2, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \text{APreOdd}_2(n-1, X_n, \ldots, Y_1, X_1, Y_0) \end{array}\right].$$

We now consider the parity function $p+1 : S \mapsto [1..2n]$, and observe that the above formula is same as the dual almost-expression for case 2. By inductive hypothesis on the dual almost-expression we have $Z^* \subseteq \{s \in S \mid \forall \pi_1 \in \Pi_1^M . \exists \pi_2 \in \Pi_2 . \Pr_s^{\pi_1,\pi_2}(\text{coParity}(p)) = 1\}$ (since $\text{Parity}(p+1) = \text{coParity}(p)$). Hence the desired claim follows.

The result follows from the above case analysis. ∎

Correctness of step 2. We now prove correctness of step 2 and we will rely on the correctness of step 1 and the inductive hypothesis. Since correctness of step 1 follows from the inductive hypothesis, we obtain the correctness of step 2 from the inductive hypothesis.

Lemma 5 *For a parity function $p : S \mapsto [1..2n+1]$, and for all $T \subseteq S$ we have $W \subseteq \mathit{Almost}_1(U, M, \mathit{Parity}(p) \cup \Diamond T)$, where W is defined as follows:*

$$\nu Y_n.\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_0.\mu X_0 \left[\begin{array}{c} T \\ \cup \\ B_{2n+1} \cap \mathit{APreOdd}_1(0, Y_n, X_n) \\ \cup \\ B_{2n} \cap \mathit{APreEven}_1(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \mathit{APreOdd}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \mathit{APreEven}_1(1, Y_n, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \mathit{APreOdd}_1(2, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \mathit{APreEven}_1(n-1, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \mathit{APreOdd}_1(n, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{array}\right]$$

Proof. We first present an intuitive explanation about the how the μ-calculus formula is obtained.

Intuitive explanation of the μ-calculus formula. The μ-calculus expression is obtained from the almost-expression for case 2: we add a $\nu Y_n.\mu X_n$ (adding a quantifier alternation of the μ-calculus formula), and every APreOdd and APreEven predecessor operators are modified by adding $\text{Apre}_1(Y_n, X_n) \uplus$ with the respective predecessor operators, and we add $B_{2n+1} \cap \text{APreOdd}_1(0, Y_n, X_n)$.

We first reformulate the algorithm for computing W in an equivalent form.

$$W = \mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_0.\mu X_0 \begin{bmatrix} T \\ \cup \\ B_{2n+1} \cap \text{APreOdd}_1(0, W, X_n) \\ \cup \\ B_{2n} \cap \text{APreEven}_1(0, W, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{APreOdd}_1(1, W, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{APreEven}_1(1, W, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{APreOdd}_1(2, W, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \text{APreEven}_1(n-1, W, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{APreOdd}_1(n, W, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{bmatrix}.$$

The reformulation is obtained as follows: since W is the fixpoint of Y_n we replace Y_n by W everywhere in the μ-calculus formula. The above mu-calculus formula is a least fixpoint and thus computes W as the limit of a sequence of sets $W_0 = T$, W_1, W_2, $\ldots$. At each iteration, both states in B_{2n+1} and states satisfying $B_{\leq 2n}$ can be added. The fact that both types of states can be added complicates the analysis of the algorithm. To simplify the correctness proof, we formulate an alternative algorithm for the computation of W; an iteration will add either a single B_{2n+1} state, or a set of $B_{\leq 2n}$ states.

To obtain the simpler algorithm, notice that the set of variables $Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0$ does not appear as an argument of the $\text{APreOdd}_1(0, W, X_n) = \text{Apre}_1(W, X_n)$ operator. Hence, each B_{2n+1}-state can be added without regards to $B_{\leq 2n}$-states that are not already in W. Moreover, since the $\nu Y_{n-1}.\mu X_{n-1}.\ldots \nu Y_0.\mu X_0$ operator applies only to $B_{\leq 2n}$-states, B_{2n+1}-states can be added one at a time. From these considerations, we can reformulate the algorithm for the computation of W as follows.

The algorithm computes W as an increasing sequence $T = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_m = W$ of states, where $m \geq 0$. Let $L_i = T_i \setminus T_{i-1}$ and the sequence is computed by computing T_i as follows, for $0 < i \leq m$:

1. either the set $L_i = \{s\}$ is a singleton such that $s \in \text{Apre}_1(W, T_{i-1}) \cap B_{2n+1}$.
2. or the set L_i consists of states in $B_{\leq 2n}$ such that L_i is a subset of the following expression

$$\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_0.\mu X_0 \begin{bmatrix} B_{2n} \cap \text{APreEven}_1(0, W, T_{i-1}, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{APreOdd}_1(1, W, T_{i-1}, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{APreEven}_1(1, W, T_{i-1}, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{APreOdd}_1(2, W, T_{i-1}, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \text{APreEven}_1(n-1, W, T_{i-1}, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{APreOdd}_1(n, W, T_{i-1}, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{bmatrix}$$

The proof that $W \subseteq \mathit{Almost}_1(U, M, \text{Parity}(p) \cup \Diamond T)$ is based on an induction on the sequence $T = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_m = W$. For $1 \leq i \leq m$, let $V_i = W \setminus T_{m-i}$, so that V_1 consists of the last block of states that has been added, V_2 to the two last blocks, and so on until $V_m = W$. We prove by induction on $i \in \{1, \ldots, m\}$, from $i = 1$ to $i = m$, that for all $s \in V_i$, there exists a uniform memoryless strategy π_1 for player 1 such that for all $\pi_2 \in \Pi_2$ we have

$$\text{Pr}_s^{\pi_1,\pi_2}\big(\Diamond T_{m-i} \cup \text{Parity}(p)\big) = 1.$$

Since the base case is a simplified version of the induction step, we focus on the latter. There are two cases, depending on whether $V_i \setminus V_{i-1}$ is composed of B_{2n+1} or of $B_{\leq 2n}$-states. Also it will follow from Lemma 11 that there always exists uniform distribution to witness that a state satisfy the required predecessor operator.

- If $V_i \setminus V_{i-1} \subseteq B_{2n+1}$, then $V_i \setminus V_{i-1} = \{s\}$ for some $s \in S$ and $s \in \text{Apre}_1(W, T_{m-i})$. The result then follows from the application of the basic Apre principle (Lemma 1) with $Z = W$, $X = T_{m-i}$, $Z \setminus Y = V_{i-1}$ and $\mathcal{A} = \text{Parity}(p)$.

- If $V_i \setminus V_{i-1} \subseteq B_{\leq 2n}$, then we analyze the predecessor operator that $s \in V_i \setminus V_{i-1}$ satisfies. The predecessor operator are essentially the predecessor operator of the almost-expression for case 2 modified by the addition of the operator $\text{Apre}_1(W, T_{m-i}) \uplus$. If player 2 plays such the $\text{Apre}_1(W, T_{m-i})$ part of the predecessor operator gets satisfied, then the analysis reduces to the previous case, and player 1 can ensure that T_{m-i} is reached with probability 1. Once we rule out the possibility of $\text{Apre}_1(W, T_{m-i})$, then the μ-calculus expression simplifies to the almost-expression of case 2, i.e.,

$$\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_0.\mu X_0 \begin{bmatrix} B_{2n} \cap \text{Pre}_1(Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{APreEven}_1(0, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \text{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{bmatrix}$$

  This ensures that if we rule out $\text{Apre}_1(W, T_{m-i})$ form the predecessor operators, then by inductive hypothesis (almost-expression for case 2) we have $L_i \subseteq \mathit{Almost}_1(U, M, \text{Parity}(p))$, and if $\text{Apre}_1(W, T_{m-i})$ is satisfied then T_{m-i} is ensured to reach with probability 1. Hence player 1 can ensure that for all $s \in V_i$, there is a uniform memoryless strategy π_1 for player 1 such that for all π_2 for player 2 we have

$$\text{Pr}_s^{\pi_1,\pi_2}\big(\Diamond T_{m-i} \cup \text{Parity}(p)\big) = 1.$$

This completes the inductive proof. With $i = m$ we obtain that there exists a uniform memoryless strategy π_1 such that for all states $s \in V_m = W$ and for all π_2 we have $\text{Pr}_s^{\pi_1,\pi_2}(\Diamond T_0 \cup \text{Parity}(p)) = 1$. Since $T_0 = T$, the desired result follows. ∎

Lemma 6 *For a parity function $p : S \mapsto [1..2n+1]$ we have $Z \subseteq \neg \mathit{Almost}_1(U, M, \mathit{Parity}(p))$, where Z is defined as follows:*

$$\mu Y_n.\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_0.\nu X_0 \left[\begin{array}{c} B_{2n+1} \cap \mathit{PosPreOdd}_2(0, Y_n, X_n) \\ \cup \\ B_{2n} \cap \mathit{PosPreEven}_2(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \mathit{PosPreOdd}_2(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \mathit{PosPreEven}_2(1, Y_n, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \mathit{PosPreOdd}_2(2, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \mathit{PosPreEven}_2(n-1, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \mathit{PosPreOdd}_2(n, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{array} \right]$$

Proof. For $k \geq 0$, let Z_k be the set of states of level k in the above μ-calculus expression. We will show that in Z_k player 2 can ensure that either Z_{k-1} is reached with positive probability or else coParity(p) is satisfied with probability 1. Since $Z_0 = \emptyset$, it would follow by induction that $Z_k \cap \mathit{Almost}_1(U, M, \text{Parity}(p)) = \emptyset$ and the desired result will follow.

We obtain of Z_k from Z_{k-1} as follows:

$$\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_0.\nu X_0 \left[\begin{array}{c} B_{2n+1} \cap \text{PosPreOdd}_2(0, Z_{k-1}, X_n) \\ \cup \\ B_{2n} \cap \text{PosPreEven}_2(0, Z_{k-1}, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{PosPreOdd}_2(1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{PosPreEven}_2(1, Z_{k-1}, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{PosPreOdd}_2(2, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \text{PosPreEven}_2(n-1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{PosPreOdd}_2(n, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{array} \right]$$

If player 1 risks into moving to Z_{k-1} with positive probability, then the inductive case is proved as Z_{k-1} is reached with positive probability. If the probability of reaching to Z_{k-1} is not positive, then the following conditions hold:

- If the probability to reach Z_{k-1} is not positive, then the predicate $\text{Pospre}_2(Z_{k-1})$ vanishes from the predecessor operator $\text{PosPreOdd}_2(i, Z_{k-1}, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i})$, and thus the operator simplifies to the simpler predecessor operator $\text{APreEven}_2(i-1, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i})$.

- If the probability to reach Z_{k-1} is not positive, then the $\text{Pospre}_2(Z_{k-1})$ vanishes from the predecessor operator $\text{PosPreEven}_2(i, Z_{k-1}, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$, and thus the operator simplifies to the predecessor operator $\text{APreOdd}_2(i, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$.

Hence either the probability to reach Z_{k-1} is positive, or the probability to reach Z_{k-1} is not positive, then the above μ-calculus expression simplifies to

$$Z^* = \nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_0.\nu X_0 \begin{bmatrix} B_{2n+1} \cap \text{Pre}_2(X_n) \\ \cup \\ B_{2n} \cap \text{APreOdd}_2(0, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{APreEven}_2(0, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{APreOdd}_2(1, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{APreEven}_2(1, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \text{APreOdd}_2(n-1, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{APreEven}_2(n-1, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{bmatrix}.$$

We now consider the parity function $p-1 : S \mapsto [0..2n]$ and by the correctness of the dual almost-expression for step 1 (Lemma 3) (with the roles of player 1 and player 2 exchanged and player 2 plays against memoryless strategies for player 1, as in Lemma 4) we have $Z^* \subseteq \{s \in S \mid \forall \pi_1 \in \Pi_1^M.\exists \pi_2 \in \Pi_2.\ \Pr_s^{\pi_1,\pi_2}(\text{coParity}(p)) = 1\}$ (since $\text{coParity}(p) = \text{Parity}(p-1)$). Hence the result follows. ∎

Correctness of step 3. The correctness of step 3 is similar to correctness of step 1. Below we present the proof sketches (since they are similar to step 1).

Lemma 7 *For a parity function $p : S \mapsto [1..2n+2]$, and for all $T \subseteq S$, we have $W \subseteq$ Almost$_1(U, M, \mathit{Parity}(p) \cup \Diamond T)$, where W is defined as follows:*

$$\nu Y_n.\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_0.\mu X_0 \begin{bmatrix} T \\ \cup \\ B_{2n+2} \cap \mathit{Pre}_1(Y_n) \\ \cup \\ B_{2n+1} \cap \mathit{APreOdd}_1(0, Y_n, X_n) \\ \cup \\ B_{2n} \cap \mathit{APreEven}_1(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \mathit{APreOdd}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \mathit{APreEven}_1(1, Y_n, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \mathit{APreOdd}_1(2, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \mathit{APreEven}_1(n-1, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \mathit{APreOdd}_1(n, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{bmatrix}$$

Proof. The proof is almost identical to the proof of Lemma 3. Similar to step 1 (Lemma 3), we add a max even priority. The proof of the result is essentially identical to the proof of Lemma 3 (almost copy-paste of the proof), the only modification is instead of the correctness of the almost-expression of case 1 we need to consider the correctness of the almost-expression for step 2 (i.e., Lemma 5 for parity function $p : S \mapsto [1..2n + 1]$). ∎

Lemma 8 *For a parity function $p : S \mapsto [1..2n + 2]$ we have $Z \subseteq \neg \mathit{Almost}_1(U, M, \mathit{Parity}(p))$, where Z is defined as follows:*

$$\mu Y_n.\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_0.\nu X_0 \left[\begin{array}{c} B_{2n+2} \cap \mathit{Pospre}_2(Y_n) \\ \cup \\ B_{2n+1} \cap \mathit{PosPreOdd}_2(0, Y_n, X_n) \\ \cup \\ B_{2n} \cap \mathit{PosPreEven}_2(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \mathit{PosPreOdd}_2(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \mathit{PosPreEven}_2(1, Y_n, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \mathit{PosPreOdd}_2(2, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \mathit{PosPreEven}_2(n-1, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \mathit{PosPreOdd}_2(n, Y_n, X_n, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{array}\right]$$

Proof. The proof of the result is identical to the proof of Lemma 4 (almost copy-paste of the proof), the only modification is instead of the correctness of the almost-expression of case 2 we need to consider the correctness of the almost-expression for step 1 (i.e., Lemma 3). This is because in the proof, after we rule out states in B_{2n+2} and analyze the sub-formula as in Lemma 3, we consider parity function $p-1 : S \mapsto [0..2n]$ and then invoke the correctness of Lemma 3. ∎

Correctness of step 4. The correctness of step 4 is similar to correctness of step 2. Below we present the proof sketches (since they are similar to step 2).

Lemma 9 *For a parity function $p : S \mapsto [0..2n + 1]$, and for all $T \subseteq S$, we have $W \subseteq$*

Almost$_1(U, M, \textit{Parity}(p) \cup \Diamond T)$*, where* W *is defined as follows:*

$$\nu Y_{n+1}.\mu X_{n+1}.\cdots \nu Y_1.\mu X_1.\nu Y_0. \left[\begin{array}{c} T \\ \cup \\ B_{2n+1} \cap \textit{APreOdd}_1(0, Y_{n+1}, X_{n+1}) \\ \cup \\ B_{2n} \cap \textit{APreEven}_1(0, Y_{n+1}, X_{n+1}, Y_n) \\ \cup \\ B_{2n-1} \cap \textit{APreOdd}_1(1, Y_{n+1}, X_{n+1}, Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \textit{APreEven}_1(1, Y_{n+1}, X_{n+1}, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \textit{APreOdd}_1(2, Y_{n+1}, X_{n+1}, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \textit{APreEven}_1(2, Y_{n+1}, X_{n+1}, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \textit{APreOdd}_1(n, Y_{n+1}, X_{n+1}, Y_n, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \textit{APreEven}_1(n, Y_{n+1}, X_{n+1}, Y_n, X_n, \ldots, Y_1, X_1, Y_0) \end{array} \right]$$

Proof. Similar to step 2 (Lemma 5), we add a max odd priority. The proof of the result is essentially identical to the proof of Lemma 5 (almost copy-paste of the proof), the only modification is instead of the correctness of the almost-expression of case 2 we need to consider the correctness of the almost-expression for step 1 (i.e., Lemma 3 for parity function $p : S \mapsto [0..2n]$). ∎

Lemma 10 *For a parity function* $p : S \mapsto [0..2n+1]$ *we have* $Z \subseteq \neg\textit{Almost}_1(U, M, \textit{Parity}(p))$*, where* Z *is defined as follows:*

$$\mu Y_{n+1}.\nu X_{n+1}.\cdots \mu Y_1.\nu X_1.\mu Y_0. \left[\begin{array}{c} B_{2n+1} \cap \textit{PosPreOdd}_2(0, Y_{n+1}, X_{n+1}) \\ \cup \\ B_{2n} \cap \textit{PosPreEven}_2(0, Y_{n+1}, X_{n+1}, Y_n) \\ \cup \\ B_{2n-1} \cap \textit{PosPreOdd}_2(1, Y_{n+1}, X_{n+1}, Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \textit{PosPreEven}_2(1, Y_{n+1}, X_{n+1}, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \textit{PosPreOdd}_2(2, Y_{n+1}, X_{n+1}, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \textit{PosPreEven}_2(2, Y_{n+1}, X_{n+1}, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \textit{PosPreOdd}_2(n, Y_{n+1}, X_{n+1}, Y_n, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \textit{PosPreEven}_2(n, Y_{n+1}, X_{n+1}, Y_n, X_n, \ldots, Y_1, X_1, Y_0) \end{array} \right]$$

Proof. The proof of the result is identical to the proof of Lemma 6 (almost copy-paste of the proof), the only modification is instead of the correctness of the almost-expression of step 1 (Lemma 3) we need to consider the correctness of the almost-expression for step 3 (i.e., Lemma 7). This is because in the proof, while we analyze the sub-formula as in Lemma 7, we consider parity function $p+1 : S \mapsto [1..2n+2]$ and then invoke the correctness of Lemma 7. ∎

Observe that above we presented the correctness for the almost-expressions for case 1 and case 2, and the correctness proofs for the dual almost-expressions are identical. We now present the duality of the predecessor operators. We first present some notations required for the proof.

Destination or possible successors of moves and distributions. Given a state s and distributions $\xi_1 \in \chi_1^s$ and $\xi_2 \in \chi_2^s$ we denote by $Dest(s,\xi_1,\xi_2) = \{t \in S \mid P_2^{\xi_1,\xi_2}(t) > 0\}$ the set of states that have positive probability of transition from s when the players play ξ_1 and ξ_2 at s. For actions a and b we have $Dest(s,a,b) = \{t \in S \mid \delta(s,a,b)(t) > 0\}$ as the set of possible successors given a and b. For $A \subseteq \Gamma_1(s)$ and $B \subseteq \Gamma_2(s)$ we have $Dest(s,A,B) = \bigcup_{a \in A, b \in B} Dest(s,a,b)$.

Lemma 11 (Duality of predecessor operators). *The following assertions hold.*

1. *Given $X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n$, we have*

$$\textit{PosPreOdd}_2(i,\neg Y_n,\neg X_n,\ldots,\neg Y_{n-i},\neg X_{n-i}) = \neg \textit{APreOdd}_1(i,Y_n,X_n,\ldots,Y_{n-i},X_{n-i}).$$

2. *Given $X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i-1} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n$, we have*

$$\begin{array}{rl} \textit{PosPreEven}_2 & (i,\neg Y_n,\neg X_n,\ldots,\neg Y_{n-i},\neg X_{n-i},\neg Y_{n-i-1}) \\ = & \neg \textit{APreEven}_1(i,Y_n,X_n,\ldots,Y_{n-i},X_{n-i},Y_{n-i-1}). \end{array}$$

3. *For all $s \in S$, if $s \in \textit{APreOdd}_1(i,Y_n,X_n,\ldots,Y_{n-i},X_{n-i})$ (resp. $s \in \textit{APreEven}_1(i,Y_n,X_n,\ldots,Y_{n-i},X_{n-i},Y_{n-i-1})$), then there exists uniform distribution ξ_1 to witness that $s \in \textit{APreOdd}_1(i,Y_n,X_n,\ldots,Y_{n-i},X_{n-i})$ (resp. $s \in \textit{APreEven}_1(i,Y_n,X_n,\ldots,Y_{n-i},X_{n-i},Y_{n-i-1})$).*

Proof. We present the proof for part 1, and the proof for second part is analogous. To present the proof of the part 1, we first present the proof for the case when $n = 2$ and $i = 2$. This proof already has all the ingredients of the general proof. After presenting the proof we present the general case.

Claim. We show that for $X_1 \subseteq X_0 \subseteq Y_0 \subseteq Y_1$ we have

$$\text{Pospre}_2(\neg Y_1) \uplus_{*} \text{Apre}_2(\neg X_1,\neg Y_0) \uplus_{*} \text{Pre}_2(\neg X_0) = \neg(\text{Apre}_1(Y_1,X_1) \uplus_{*} \text{Apre}_1(Y_0,X_0)).$$

We now present the following two case analysis for the proof.

1. A subset $U \subseteq \Gamma_1(s)$ is *good* if both the following conditions hold:
   (a) *Condition 1.* For all $b \in \Gamma_2(s)$ and for all $a \in U$ we have $Dest(s,a,b) \subseteq Y_1$ (i.e., $Dest(s,U,b) \subseteq Y_1$); and
   (b) *Condition 2.* For all $b \in \Gamma_2(s)$ one of the following conditions hold:
      i. either there exists $a \in U$ such that $Dest(s,a,b) \cap X_1 \neq \emptyset$ (i.e., $Dest(s,U,b) \cap X_1 \neq \emptyset$); or
      ii. for all $a \in U$ we have $Dest(s,a,b) \subseteq Y_0$ (i.e., $Dest(s,U,b) \subseteq Y_0$) and for some $a \in U$ we have $Dest(s,a,b) \cap X_0 \neq \emptyset$ (i.e., $Dest(a,U,b) \cap X_0 \neq \emptyset$).

We show that if there is a good set U, then $s \in \text{Apre}_1(Y_1, X_1) \mathbin{\cup\!\!\!\ast} \text{Apre}_1(Y_0, X_0)$. Given a good set U, consider the *uniform* distribution ξ_1 that plays all actions in U uniformly at random. Consider an action $b \in \Gamma_2(s)$ and the following assertions hold:

(a) By condition 1 we have $Dest(s, \xi_1, b) \subseteq Y_1$.

(b) By condition 2 we have either (i) $Dest(s, \xi_1, b) \cap X_1 \neq \emptyset$ (if condition 2.a holds); or (ii) $Dest(s, \xi_1, b) \subseteq Y_0$, and $Dest(s, \xi_1, b) \cap X_0 \neq \emptyset$ (if condition 2.b holds).

It follows that in all cases we have (i) either $Dest(s, \xi_1, b) \subseteq Y_1$ and $Dest(s, \xi_1, b) \cap X_1 \neq \emptyset$, or (ii) $Dest(s, \xi_1, b) \subseteq Y_0$ and $Dest(s, \xi_1, b) \cap X_0 \neq \emptyset$. It follows that ξ_1 is a uniform distribution witness to show that $s \in \text{Apre}_1(Y_1, X_1) \mathbin{\cup\!\!\!\ast} \text{Apre}_1(Y_0, X_0)$.

2. We now show that if there is no good set U, then $s \in \text{Pospre}_2(\neg Y_1) \mathbin{\cup\!\!\!\ast} \text{Apre}_2(\neg X_1, \neg Y_0) \mathbin{\cup\!\!\!\ast} \text{Pre}_2(\neg X_0)$. Given a set U, if U is not good, then (by simple complementation argument) one of the following conditions must hold:

   (a) *Complementary Condition 1.* There exists $b \in \Gamma_2(s)$ and $a \in U$ such that $Dest(s, a, b) \cap \neg Y_1 \neq \emptyset$; or

   (b) *Complementary Condition 2.* There exists $b \in \Gamma_2(s)$ such that both the following conditions hold:

      i. for all $a \in U$ we have $Dest(s, a, b) \subseteq \neg X_1$; and

      ii. there exists $a \in U$ such that $Dest(s, a, b) \cap \neg Y_0 \neq \emptyset$ or for all $a \in U$ we have $Dest(s, a, b) \subseteq \neg X_0$.

   Since there is no good set, for every set $U \subseteq \Gamma_1(s)$, there is a counter action $b = c(U) \in \Gamma_2(s)$, such that one of the complementary conditions hold. Consider a distribution ξ_1 for player 1, and let $U = \text{Supp}(\xi_1)$. Since U is not a good set, consider a counter action $b = c(U)$ satisfying the complementary conditions. We now consider the following cases:

   (a) If complementary condition 1 holds, then $Dest(s, \xi_1, b) \cap \neg Y_1 \neq \emptyset$ (i.e., $\text{Pospre}_2(\neg Y_1)$ is satisfied).

   (b) Otherwise complementary condition 2 holds, and by 2.a we have $Dest(s, \xi_1, b) \subseteq \neg X_1$.

      i. if there exists $a \in U$ such that $Dest(s, a, b) \cap \neg Y_0 \neq \emptyset$, then $Dest(s, \xi_1, b) \cap \neg Y_0 \neq \emptyset$ (hence $\text{Apre}_2(\neg X_1, \neg Y_0)$ holds);

      ii. otherwise for all $a \in U$ we have $Dest(s, a, b) \subseteq \neg X_0$, hence $Dest(s, \xi_1, b) \subseteq \neg X_0$ (hence $\text{Pre}_2(\neg X_0)$ holds).

   The claim follows.

General case. We now present the result for the general case which is a generalization of the previous case. We present the details here, and will omit it in later proofs, where the argument is similar. Recall that we have the following inclusion: $X_n \subseteq X_{n-1} \subseteq \ldots \subseteq X_{n-i} \subseteq Y_{n-i} \subseteq \ldots Y_{n-1} \subseteq Y_n$.

1. A subset $U \subseteq \Gamma_1(s)$ is *good* if both the following conditions hold: for all $b \in \Gamma_2(s)$

   (a) *Condition 1.* For all $a \in U$ we have $Dest(s, a, b) \subseteq Y_n$ (i.e., $Dest(s, U, b) \subseteq Y_n$); and

(b) *Condition 2.* There exists $0 \leq j \leq i$, such that for all $a \in U$ we have $Dest(s,a,b) \subseteq Y_{n-j}$ (i.e., $Dest(s,U,b) \subseteq Y_{n-j}$), and for some $a \in U$ we have $Dest(s,a,b) \cap X_{n-j} \neq \emptyset$ (i.e., $Dest(s,U,b) \cap X_{n-j} \neq \emptyset$).

We show that if there is a good set U, then $s \in \text{Apre}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$. Given a good set U, consider the ***uniform*** distribution ξ_1 that plays all actions in U uniformly at random. Consider an action $b \in \Gamma_2(s)$ and the following assertions hold:

(a) By condition 1 we have $Dest(s,\xi_1,b) \subseteq Y_n$.

(b) By condition 2 we have for some $0 \leq j \leq i$, we have $Dest(s,\xi_1,b) \subseteq Y_{n-j}$, and $Dest(s,\xi_1,b) \cap X_{n-j} \neq \emptyset$ (i.e., $\text{Apre}_1(Y_{n-j}, X_{n-j})$ holds).

It follows that ξ_1 is a uniform distribution witness to show that $s \in \text{Apre}_1(Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$.

2. We now show that if there is no good set U, then $s \in \text{PosPreOdd}_2(i, \neg Y_n, \neg X_n, \ldots, \neg Y_{n-i}, \neg X_{n-i})$. Given a set U, if U is not good, then we show that one of the following conditions must hold: there exists $b \in \Gamma_2(s)$ such that

   (a) *Complementary Condition 1 (CC1).* $Dest(s,U,b) \cap \neg Y_n \neq \emptyset$; or

   (b) *Complementary Condition 2 (CC2).* there exists $0 \leq j < i$ such that $Dest(s,U,b) \subseteq \neg X_{n-j}$ and $Dest(s,U,b) \cap \neg Y_{n-j-1} \neq \emptyset$; or

   (c) *Complementary Condition 3 (CC3).* $Dest(s,U,b) \subseteq \neg X_{n-i}$.

   Consider a set U that is not good, and let b be an action that witness that U is not good. We show that b satisfies one of the complementary conditions.

   - If $Dest(s,U,b) \cap \neg Y_n \neq \emptyset$, then we are done as CC1 is satisfied. Otherwise, we have $Dest(s,U,b) \subseteq Y_n$, then we must have $Dest(s,U,b) \subseteq \neg X_n$ (otherwise the action b would satisfy the condition $Dest(s,U,b) \subseteq Y_n$ and $Dest(s,U,b) \cap X_n \neq \emptyset$, and cannot be a witness that U is not good). Now we continue: if $Dest(s,U,b) \cap \neg Y_{n-1} \neq \emptyset$, then we are done, as we have a witness that $Dest(s,U,b) \subseteq \neg X_n$ and $Dest(s,U,b) \cap \neg Y_{n-1} \neq \emptyset$. If $Dest(s,U,b) \subseteq Y_{n-1}$, then again since b is witness to show that U is not good, we must have $Dest(s,U,b) \subseteq \neg X_{n-1}$. We again continue, and if we have $Dest(s,U,b) \cap \neg Y_{n-2} \neq \emptyset$, we are done, or else we continue and so on. Thus we either find a witness $0 \leq j < i$ to satisfy CC2, or else in the end we have that $Dest(s,U,b) \subseteq \neg X_{n-i}$ (satisfies CC3).

   Since there is no good set, for every set $U \subseteq \Gamma_1(s)$, there is a counter action $b = c(U) \in \Gamma_2(s)$, such that one of the complementary conditions hold. Consider a distribution ξ_1 for player 1, and let $U = \text{Supp}(\xi_1)$. Since U is not a good set, consider a counter action $b = c(U)$ satisfying the complementary conditions. We now consider the following cases:

   (a) If CC1 1 holds, then $Dest(s,U,b) \cap \neg Y_n \neq \emptyset$ (hence also $Dest(s,\xi_1,b) \cap \neg Y_n \neq \emptyset$) (i.e., $\text{Pospre}_2(\neg Y_n)$ is satisfied).

   (b) Else if CC2 holds, then for some $0 \leq j < i$ and we have $Dest(s,U,b) \subseteq \neg X_{n-j}$ and $Dest(s,U,b) \subseteq Y_{n-j-1}$ (hence also $Dest(s,\xi_1,b) \subseteq \neg X_{n-j}$ and $Dest(s,\xi_1,b) \subseteq Y_{n-j-1}$) (i.e., $\text{Apre}_2(\neg X_n, \neg Y_{n-1}) \uplus \text{Apre}_2(\neg X_{n-1}, \neg Y_{n-2}) \uplus \ldots \uplus \text{Apre}_2(\neg X_{n-i+1}, \neg Y_{n-i})$ holds).

   (c) Otherwise CC3 holds and we have $Dest(s,U,b) \subseteq \neg X_{n-i}$, (hence also $Dest(s,\xi_1,b) \subseteq \neg X_{n-i}$) (i.e., $\text{Pre}_2(\neg X_{n-i})$ holds).

The claim follows.

The result for part 3 follows as in the above proofs we have always constructed uniform witness distribution. ∎

Characterization of *Almost*$_1(U, M, \Phi)$ set. From Lemmas 3—10, and the duality of predecessor operators (Lemma 11) we obtain the following result characterizing the almost-sure winning set for uniform memoryless strategies for parity objectives.

Theorem 3 *For all concurrent game structures $\mathcal{G}$ over state space S, for all parity objectives Parity(p) for player 1, the following assertions hold.*

1. *If $p : S \mapsto [0..2n-1]$, then Almost$_1(U, M, \mathit{Parity}(p)) = W$, where W is defined as follows*

$$\nu Y_n.\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0. \left[\begin{array}{c} B_{2n-1} \cap \mathit{APreOdd}_1(0, Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \mathit{APreEven}_1(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \mathit{APreOdd}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \mathit{APreEven}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \mathit{APreOdd}_1(n-1, Y_n, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_0 \cap \mathit{APreEven}_1(n-1, Y_n, X_n, \ldots, Y_1, X_1, Y_0) \end{array} \right] \tag{1}$$

   and $B_i = p^{-1}(i)$ is the set of states with priority i, for $i \in [0..2n-1]$.

2. *If $p : S \mapsto [1..2n]$, then Almost$_1(U, M, \mathit{Parity}(p)) = W$, where W is defined as follows*

$$\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0.\mu X_0 \left[\begin{array}{c} B_{2n} \cap \mathit{Pre}_1(Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \mathit{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \mathit{APreEven}_1(0, Y_{n-1}, X_{n-2}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \mathit{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \vdots \\ B_2 \cap \mathit{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \mathit{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{array} \right] \tag{2}$$

   and $B_i = p^{-1}(i)$ is the set of states with priority i, for $i \in [1..2n]$.

3. *The set Almost$_1(U, M, \mathit{Parity}(p))$ can be computed symbolically using the expressions (1) and (2) in time $\mathcal{O}(|S|^{2n+1} \cdot \sum_{s \in S} 2^{|\Gamma_1(s) \cup \Gamma_2(s)|})$.*

4. *Given a state $s \in S$ whether $s \in$ Almost$_1(U, M, \mathit{Parity}(p))$ can be decided in NP $\cap$ coNP.*

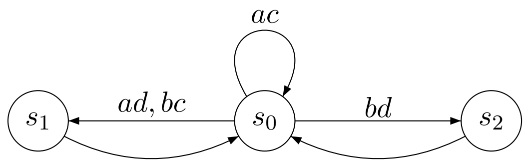

Figure 1: Three priority concurrent game

Ranking function for μ-calculus formula. Given a μ-calculus formula of alternation-depth (the nesting depth of ν-μ-operators), the *ranking* function maps every state to a tuple of d-integers, such that each integer is at most the size of the state space. For a state that satisfies the μ-calculus formula the tuple of integers denote iterations of the μ-calculus formula such that the state got included for the first time in the nested evaluation of the μ-calculus formula (for details see [EJ91, Koz83]).

The NP $\cap$ coNP bound follows directly from the μ-calculus expressions as the players can guess the *ranking function* of the μ-calculus formula and the support of the uniform distribution at every state to witness that the predecessor operator is satisfied, and the guess can be verified in polynomial time. Observe that the computation through μ-calculus formulas is symbolic and more efficient than enumeration over the set of all uniform memoryless strategies of size $O(\prod_{s \in S} |\Gamma_1(s) \cup \Gamma_2(s)|)$ (for example, with constant action size and constant d, the μ-calculus formula is polynomial, whereas enumeration of strategies is exponential). The μ-calculus formulas of [EJ91] can be obtained as a special case of the μ-calculus formula of Theorem 3 by replacing all predecessor operators with the Pre_1 predecessor operator.

Proposition 3 $\mathit{Almost}_1(IP, FM, \Phi) = \mathit{Almost}_1(U, FM, \Phi) = \mathit{Almost}_1(U, M, \Phi)$.

Proof. Consider a finite-memory strategy that is almost-sure winning. Since it is finite-memory, it must be finite-precision. The result follows from Proposition 2. ∎

It follows from above that uniform memoryless strategies are as powerful as finite-precision infinite-memory strategies for almost-sure winning. We now show that infinite-precision infinite-memory strategies are more powerful than uniform memoryless strategies.

Example 2 (*Almost*$_1(U, M, \Phi) \subsetneq$ *Almost*$_1(IP, IM, \Phi)$). We show with an example that for a concurrent parity game with three priorities we have $\mathit{Almost}_1(U, M, \Phi) \subsetneq \mathit{Almost}_1(IP, IM, \Phi)$. Consider the game shown in Fig 1. The moves available for player 1 and player 2 at s_0 is $\{a, b\}$ and $\{c, d\}$, respectively. The priorities are as follows: $p(s_0) = 1, p(s_2) = 3$ and $p(s_1) = 2$. In other words, player 1 wins if s_1 is visited infinitely often and s_2 is visited finitely often. We show that for all uniform memoryless strategy for player 1 there is counter strategy for player 2 to ensure that the co-parity condition is satisfied with probability 1. Consider a memoryless strategy π_1 for player 1, and the counter strategy π_2 is defined as follows: (i) if $b \in \mathrm{Supp}(\pi_1(s_0))$, then play d, (ii) otherwise, play c. It follows that (i) if $b \in \mathrm{Supp}(\pi_1(s_0))$, then the closed recurrent set C of the Markov chain obtained by fixing π_1 and π_2 contains s_2, and hence s_2 is visited infinitely often with probability 1; (ii) otherwise, player 1 plays the deterministic memoryless strategy that plays a at s_0, and the counter move c ensures that only s_0 is visited infinitely often. It follows from our results that for all finite-memory strategies for player 1, player 2 can ensure that player 1 cannot win with probability 1.

We now show that in the game there is an infinite-memory infinite-precision strategy for player 1 to win with probability 1 against all player 2 strategies. Consider a strategy π_1 for player 1 that is played in rounds,

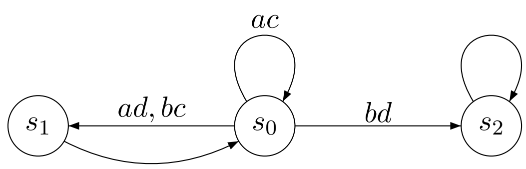

Figure 2: Büchi games

and a round is incremented upon visit to $\{s_1, s_2\}$, and in round k the strategy plays action a with probability $1 - \frac{1}{2^{k+1}}$ and b with probability $\frac{1}{2^{k+1}}$. For $k \geq 0$, let $\mathcal{E}_k$ denote the event that the game gets stuck at round k. In round k, against any strategy for player 2 in every step there is at least probability $\eta_k = \frac{1}{2^{k+1}} > 0$ to visit the set $\{s_1, s_2\}$. Thus the probability to be in round k for ℓ steps is at most $(1 - \eta_k)^\ell$, and this is 0 as ℓ goes to ∞. Thus we have $\mathrm{Pr}_{s_0}^{\pi_1,\pi_2}(\mathcal{E}_k) = 0$. Hence the probability that the game is stuck in some round k is

$$\mathrm{Pr}_{s_0}^{\pi_1,\pi_2}(\bigcup_{k\geq 0} \mathcal{E}_k) \leq \sum_{k \geq 0} \mathrm{Pr}_{s_0}^{\pi_1,\pi_2}(\mathcal{E}_k) = 0,$$

where the last equality follows as the countable sum of probability zero event is zero. It follows that $\mathrm{Pr}_{s_0}^{\pi_1,\pi_2}(\Box\Diamond\{s_1, s_2\}) = 1$, i.e., $\{s_1, s_2\}$ is visited infinitely often with probability 1. To complete the proof we need to show that $\{s_2\}$ is visited infinitely often with probability 0. Consider an arbitrary strategy for player 2. We first obtain the probability u_{k+1} that s_2 is visited $k+1$ times, given it has been visited k times. Observe that to visit s_2 player 2 must play the action d, and thus

$$u_{k+1} \leq \frac{1}{2^{k+1}}(1 + \frac{1}{2} + \frac{1}{4} + \ldots),$$

where in the infinite sum is obtained by considering the number of consecutive visits to s_1 before s_2 is visited. The explanation of the infinite sum is as follows: the probability to reach s_2 for $k+1$-th time after the k-th visit (i) with only one visit to s_1 is $\frac{1}{2^{k+1}}$, (ii) with two visits to s_1 is $\frac{1}{2^{k+2}}$ (as the probability to play action b is halved), (iii) with three visits to s_1 is $\frac{1}{2^{k+3}}$ and so on. Hence we have $u_{k+1} \leq \frac{1}{2^k}$. The probability that s_2 is visited infinitely often is $\prod_{k=0}^{\infty} u_{k+1} \leq \prod_{k=0}^{\infty} \frac{1}{2^{k+1}} = 0$. It follows that for all strategies π_2 we have $\mathrm{Pr}_{s_0}^{\pi_1,\pi_2}(\Box\Diamond\{s_2\}) = 0$, and hence $\mathrm{Pr}_{s_0}^{\pi_1,\pi_2}(\Box\Diamond\{s_1\} \cap \Diamond\Box\{s_1, s_0\}) = 1$. Thus we have shown that player 1 has an infinite-memory infinite-precision almost-sure winning strategy. ∎

Example 3 (*Limit*$_1(IP, FM, \Phi) \subsetneq$ *Limit*$_1(IP, IM, \Phi)$). We show with an example that *Limit*$_1(IP, FM, \Phi) \subsetneq$ *Limit*$_1(IP, IM, \Phi)$. The example is from [dAH00] and we present the details for the sake of completeness.

Consider the game shown in Fig. 2. The state s_2 is an absorbing state, and from the state s_1 the next state is always s_0. The objective of player 1 is to visit s_1 infinitely often, i.e., $\Box\Diamond\{s_1\}$. For $\varepsilon > 0$, we will construct a strategy π_1^ε for player 1 that ensures s_1 is visited infinitely often with probability at least $1 - \varepsilon$. First, given $\varepsilon > 0$, we construct a sequence of ε_i, for $i \geq 0$, such that $\varepsilon_i > 0$, and $\prod_i (1 - \varepsilon_i) \geq (1 - \varepsilon)$. Let $\pi_1^{\varepsilon_i}$ be a memoryless strategy for player 1 that ensures s_0 is reached from s_1 with probability at least $1 - \varepsilon_i$; such a strategy can be constructed as in the solution of reachability games (see [dAHK07]). The strategy π_1^ε is as follows: for a history $w \in S^*$ (finite sequence of states), if the number of times s_1 has appeared in w is i, then for the history $w \cdot s_0$ the strategy π_1^ε plays like $\pi_1^{\varepsilon_i}$, i.e., $\pi_1^\varepsilon(w \cdot s_0) = \pi_1^{\varepsilon_i}(s_0)$. The strategy π_ε constructed in this fashion ensures that against any strategy π_2, the state s_1 is visited infinitely often with probability at least $\prod_i (1-\varepsilon_i) \geq 1-\varepsilon$. However, the strategy π_1^ε counts the number of visits to s_1, and therefore uses infinite memory.

We now show that the infinite memory requirement cannot be avoided. We show now that all finite-memory strategies visit s_2 infinitely often with probability 0. Let π be an arbitrary finite-memory strategy for player 1, and let M be the (finite) memory set used by the strategy. Consider the product game graph defined on the state space $\{s_0, s_1, s_2\} \times M$ as follows: for $s \in \{s_0, s_1, s_2\}$ and $m \in M$, let $\pi_u(s,m) = m_1$ (where π_u is the memory update function of π), then for $a_1 \in \Gamma_1(s)$ and $b_1 \in \Gamma_2(s)$ we have

$$\overline{\delta}((s,m), a_1, b_1)(s', m') = \begin{cases} \delta(s, a_1, b_1)(s') & m' = m_1 \\ 0 & \text{otherwise} \end{cases}$$

where $\overline{\delta}$ is the transition function of the product game graph. The strategy π will be interpreted as a memoryless $\overline{\pi}$ in the product game graph as follows: for $s \in \{s_0, s_1, s_2\}$ and $m \in M$ we have $\overline{\pi}((s,m)) = \pi_n((s,m))$, where π_n is the next move function of π. Consider now a strategy π_2 for player 2 constructed as follows. From a state $(s_0, m) \in \{s_0, s_1, s_2\} \times M$, if the strategy $\overline{\pi}$ plays a with probability 1, then player 2 plays c with probability 1, ensuring that the successor is (s_0, m') for some $m' \in M$. If $\overline{\pi}$ plays b with positive probability, then player 2 plays c and d uniformly at random, ensuring that (s_2, m') is reached with positive probability, for some $m' \in M$. Under π_1, π_2 the game is reduced to a Markov chain, and since the set $\{s_2\} \times M$ is absorbing, and since all states in $\{s_0\} \times M$ either stay safe in $\{s_0\} \times M$ or reach $\{s_2\} \times M$ in one step with positive probability, and all states in $\{s_1\} \times M$ reach $\{s_0\} \times M$ in one step, the closed recurrent classes must be either entirely contained in $\{s_0\} \times M$, or in $\{s_2\} \times M$. This shows that, under π_1, π_2, player 1 achieves the Büchi goal $\Box\Diamond\{s_1\}$ with probability 0. ∎

4 Infinite-precision Strategies

The results of the previous section already characterizes that for almost-sure winning infinite-precision finite-memory strategies are no more powerful than uniform memoryless strategies. In this section we characterize the limit-sure winning for infinite-precision finite-memory strategies. We define two new operators, Lpre (limit-pre) and Fpre (fractional-pre). For $s \in S$ and $X, Y \subseteq S$, these two-argument predecessor operators are defined as follows:

$$\mathrm{Lpre}_1(Y, X) = \{s \in S \mid \forall \alpha > 0 \,.\, \exists \xi_1 \in \chi_1^s \,.\, \forall \xi_2 \in \chi_2^s \,.\, \left[P_s^{\xi_1,\xi_2}(X) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y)\right]\}; \tag{3}$$

$$\mathrm{Fpre}_2(X, Y) = \{s \in S \mid \exists \beta > 0 \,.\, \forall \xi_1 \in \chi_1^s \,.\, \exists \xi_2 \in \chi_2^s \,.\, \left[P_s^{\xi_1,\xi_2}(Y) \geq \beta \cdot P_s^{\xi_1,\xi_2}(\neg X)\right]\} \,. \tag{4}$$

The operator $\mathrm{Lpre}_1(Y, X)$ is the set of states such that player 1 can choose distributions to ensure that the probability to progress to X can be made arbitrarily large as compared to the probability of escape from Y. In other words, the probability to progress to X divided by the sum of the probability to progress to X and to escape Y can be made arbitrarily close to 1 (in the limit 1). The operator $\mathrm{Fpre}_2(X, Y)$ is the set of states such that against all player 1 distributions, player 2 can choose a distribution to ensure that the probability to progress to Y can be made greater than a positive constant times the probability of escape from X, (i.e., progress to Y is a positive fraction of the probability to escape from X).

Limit-sure winning for memoryless strategies. The results of [dAHK07] shows that for reachability objectives, memoryless strategies suffices for limit-sure winning. We now show with an example that limit-sure winning for Büchi objectives with memoryless strategies is not simply limit-sure reachability to the set of almost-sure winning states. Consider the game shown in Fig 3 with actions $\{a, b\}$ for player 1 and

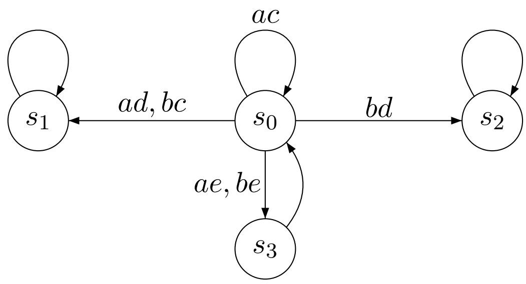

Figure 3: A Büchi game

$\{c, d, e\}$ for player 2 at s_0. States s_1, s_2 are absorbing, and from s_3 the successor is s_0 deterministically. The Büchi objective is to visit $\{s_1, s_3\}$ infinitely often. The only almost-sure winning state is $\{s_1\}$. The state s_0 is not almost-sure winning because at s_0 if player 1 plays b with positive probability the counter move is d, otherwise the counter move is c. Hence either s_2 is reached with positive probability or s_0 is never left. Moreover, player 1 cannot limit-sure reach the state s_1 from s_0, as the move e ensures that s_1 is never reached. Thus in this game the limit-sure reach to the almost-sure winning set is only state s_1. We now show that for all ε, there is a memoryless strategy to ensure the Büchi objective with probability at least $1 - \varepsilon$ from s_0. At s_0 the memoryless strategy plays a with probability $1 - \varepsilon$ and b with probability ε. Fixing the strategy for player 1 we obtain an MDP for player 2, and in the MDP player 2 has an optimal pure memoryless strategy. If player 2 plays the pure memoryless strategy e, then s_3 is visited infinitely often with probability 1; if player 2 plays the pure memoryless strategy c, then s_1 is reached with probability 1; and if player 2 plays d, then s_1 is reached with probability $1 - \varepsilon$. Thus for all $\varepsilon > 0$, player 1 can win from s_0 and s_2 with probability at least $1 - \varepsilon$ with a memoryless strategy.

Limit-winning set for Büchi objectives. We first present the characterization of the set of limit-sure winning states for concurrent Büchi games from [dAH00] for infinite-memory and infinite-precision strategies. The limit-sure winning set is characterized by the following formula

$$\nu Y_0.\mu X_0.[(B \cap \mathrm{Pre}_1(Y_0)) \cup (\neg B \cap \mathrm{Lpre}_1(Y_0, X_0))]$$

Our characterization of the limit-sure winning set for memoryless infinite-precision strategies would be obtained as follows: we will obtain sequence of chunk of states $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k$ such that from each X_i for all $\varepsilon > 0$ there is a memoryless strategy to ensure that $\Diamond X_{i-1} \cup (\Box\Diamond B \cap \Box(X_i \setminus X_{i-1}))$ is satisfied with probability at least $1 - \varepsilon$. We consider the following μ-calculus formula:

$$\nu Y_1.\mu X_1.\nu Y_0.\mu X_0.[(B \cap \mathrm{Pre}_1(Y_0) \circledast \mathrm{Lpre}_1(Y_1, X_1)) \cup (\neg B \cap \mathrm{Apre}_1(Y_0, X_0) \circledast \mathrm{Lpre}_1(Y_1, X_1))]$$

Let Y^* be the fixpoint, and since it is a fixpoint we have

$$Y^* = \mu X_1.\nu Y_0.\mu X_0. \left[\begin{array}{l} \big(B \cap \mathrm{Pre}_1(Y_0) \circledast \mathrm{Lpre}_1(Y^*, X_1)\big) \cup \\ \big(\neg B \cap \mathrm{Apre}_1(Y_0, X_0) \circledast \mathrm{Lpre}_1(Y^*, X_1)\big) \end{array} \right]$$

Hence Y^* is computed as least fixpoint as sequence of sets $X_0 \subseteq X_1 \ldots \subseteq X_k$, and X_{i+1} is obtained from X_i as

$$\nu Y_0.\mu X_0.[(B \cap \mathrm{Pre}_1(Y_0) \circledast \mathrm{Lpre}_1(Y^*, X_i)) \cup (\neg B \cap \mathrm{Apre}_1(Y_0, X_0) \circledast \mathrm{Lpre}_1(Y^*, X_i))]$$

The $\mathrm{Lpre}_i(Y^*, X_i)$ is similar to limit-sure reachability to X_i, and once we rule out $\mathrm{Lpre}_1(Y^*, X_i)$, the formula simplifies to the almost-sure winning under memoryless strategies. In other words, from each X_{i+1} player 1 can ensure with a memoryless strategy that either (i) X_i is reached with limit probability 1 or

(ii) the game stays in $X_{i+1} \setminus X_i$ and the Büchi objective is satisfied with probability 1. It follows that $Y^* \subseteq \mathit{Limit}_1(IP, M, \Box\Diamond B)$. We will show that in the complement set there exists constant $\eta > 0$ such that for all finite-memory infinite-precision strategies for player 1 there is a counter strategy to ensure the complementary objective with probability at least $\eta > 0$.

The general principle. The general principle to obtain the μ-calculus formula for limit-sure winning for memoryless infinite-precision strategies is as follows: we consider the μ-calculus formula for the almost-sure winning for uniform memoryless strategies, then add a $\nu Y_{n+1} \mu X_{n+1}$ quantifier and add the $\mathrm{Lpre}_1(Y_{n+1}, X_{n+1}) \circledast$ to every predecessor operator. Intuitively, when we replace Y_{n+1} by the fixpoint Y^*, then we obtain sequence X_i of chunks of states for the least fixpoint computation of X_{n+1}, such that from X_{i+1} either X_i is reached with limit probability 1 (by the $\mathrm{Lpre}_1(Y^*, X_{n+1})$ operator), or the game stays in $X_{i+1} \setminus X_i$ and then the parity objective is satisfied with probability 1 by a memoryless strategy. Formally, we will show Lemma 13, and we first present a technical lemma required for the correctness proof.

Lemma 12 (Basic Lpre principle). *Let $X \subseteq Y \subseteq Z \subseteq S$ and such that all $s \in Y \setminus X$ we have $s \in$ $\mathit{Lpre}_1(Z, X)$. For all prefix-independent events $\mathcal{A} \subseteq \Box(Z \setminus Y)$, the following assertion holds:*

> *Assume that for all $\eta > 0$ there exists a memoryless strategy $\pi_1^\eta \in \Pi_1^M$ such that for all $\pi_2 \in \Pi_2$ and for all $z \in Z \setminus Y$ we have*
>
> $$\mathrm{Pr}_z^{\pi_1^\eta, \pi_2}(\mathcal{A} \cup \Diamond Y) \geq 1 - \eta, \qquad (\textit{i.e.},\ \lim_{\eta \to 0} \mathrm{Pr}_z^{\pi_1^\eta, \pi_2}(\mathcal{A} \cup \Diamond Y) = 1).$$
>
> *Then, for all $s \in Y$ for all $\varepsilon > 0$ there exists a memoryless strategy $\pi_1^\varepsilon \in \Pi_1^M$ such that for all $\pi_2 \in \Pi_2$ we have*
>
> $$\mathrm{Pr}_s^{\pi_1^\varepsilon, \pi_2}(\mathcal{A} \cup \Diamond X) \geq 1 - \varepsilon, \qquad (\textit{i.e.},\ \lim_{\varepsilon \to 0} \mathrm{Pr}_s^{\pi_1^\varepsilon, \pi_2}(\mathcal{A} \cup \Diamond X) = 1).$$

Proof. The situation is depicted in Figure 4.(a). Since for all $s \in Y \setminus X$ we have $s \in \mathrm{Lpre}_1(Z, X)$, given $\varepsilon > 0$, player 1 can play the distribution $\xi_{s,1}^{\mathrm{Lpre}}[\varepsilon](Z, X)$ to ensure that the probability of going to $\neg Z$ is at most ε times the probability of going to X. Fix a counter strategy π_2 for player 2. Let γ and γ' denote the probability of going to X and $\neg Z$, respectively. Then $\gamma' \leq \varepsilon \cdot \gamma$. Observe that $\gamma > \varepsilon^l$, where $l = |\Gamma_s|$. Let α denote the probability of the event $\mathcal{A}$. We first present an informal argument and then present rigorous calculations. Since $\mathcal{A} \subseteq \mathcal{A} \cup \Diamond X$, the worst-case analysis for the result correspond to the case when $\alpha = 0$, and the simplified situation is shown as Fig 4.(b). Once we let $\eta \to 0$, then we only have an edge from $Z \setminus Y$ to Y and the situation is shown in Fig 4.(c). If q is the probability to reach X, then the probability to reach $\neg Z$ is $q \cdot \varepsilon$ and we have $q + q\varepsilon = 1$, i.e., $q = \frac{1}{1+\varepsilon}$, and given $\varepsilon' > 0$ we can chose ε to ensure that $q \geq 1 - \varepsilon'$.

We now present detailed calculations. Given $\varepsilon' > 0$ we construct a strategy $\pi_1^{\varepsilon'}$ as follows: let $\varepsilon = \frac{\varepsilon'}{2(1-\varepsilon')}$ and $\eta = \varepsilon^{l+1} > 0$; and fix the strategy π_1^η for states in $Z \setminus Y$ and the distribution $\xi_{s,1}^{\mathrm{Lpre}}[\varepsilon](Z, X)$ at s. Observe that by choice we have $\eta \leq \gamma \cdot \varepsilon$. Let $q = \mathrm{Pr}_s^{\pi_1^{\varepsilon'}, \pi_2}(\mathcal{A} \cup \Diamond X)$. Then we have $q \geq \gamma + \beta \cdot \big(\alpha + (1 - \eta - \alpha) \cdot q\big)$; since the set $Z \setminus Y$ is reached with probability at most β and then again Y is reached with probability at least $1 - \eta - \alpha$ and event $\mathcal{A}$ happens with probability at least α. Hence we have

$$q \geq \gamma + \beta \cdot \big(\alpha + (1 - \eta - \alpha) \cdot q\big) \geq \gamma + \beta \cdot \big(\alpha \cdot q + (1 - \eta - \alpha) \cdot q\big) = \gamma + \beta \cdot (1 - \eta) \cdot q;$$

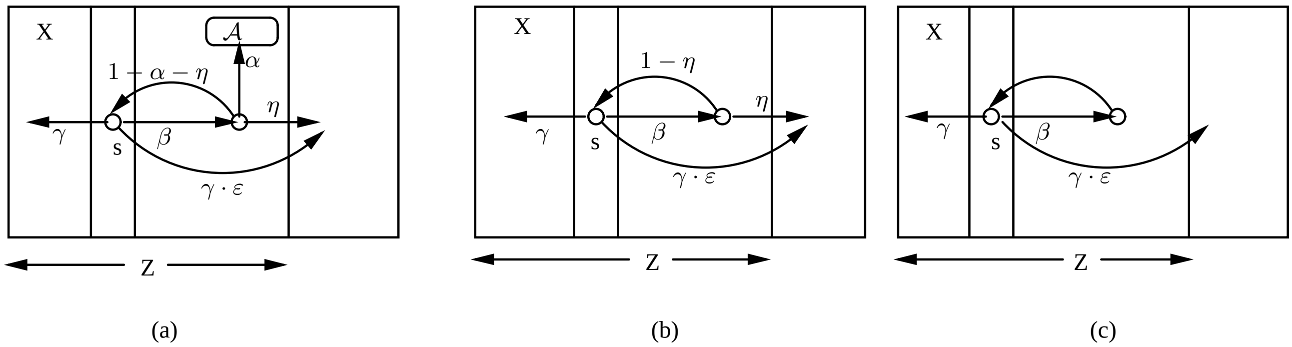

Figure 4: Basic Lpre principle; in the figures $\beta = 1-\gamma-\gamma\cdot\varepsilon$

the first inequality follows as $q \leq 1$. Thus we have

$$\begin{array}{rcll} q & \geq & \gamma + (1-\gamma-\gamma\cdot\varepsilon)\cdot(1-\eta)\cdot q; & \\ q & \geq & \dfrac{\gamma}{\gamma+\gamma\cdot\varepsilon+\eta-\eta\cdot\gamma-\eta\cdot\gamma\cdot\varepsilon} & \\ & \geq & \dfrac{\gamma}{\gamma+\gamma\cdot\varepsilon+\eta} & \\ & \geq & \dfrac{\gamma}{\gamma+\gamma\cdot\varepsilon+\gamma\cdot\varepsilon} & \text{(since } \eta \leq \gamma\cdot\varepsilon) \\ & \geq & \frac{1}{1+2\varepsilon} \geq 1-\varepsilon'. & \end{array}$$

The desired result follows. ∎

Lemma 13 *For a parity function $p : S \mapsto [1..2n]$ and $T \subseteq S$, we have $W \subseteq$ Limit$_1(IP, M,$ Parity$(p) \cup \Diamond T)$, where W is defined as follows:*

$$\nu Y_n.\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots\nu Y_1.\mu X_1.\nu Y_0.\mu X_0.$$
$$\left[\begin{array}{c} T \\ B_{2n} \cap \mathit{Pre}_1(Y_{n-1}) \uplus_{*} \mathit{Lpre}_1(Y_n, X_n) \\ \cup \\ B_{2n-1} \cap \mathit{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \uplus_{*} \mathit{Lpre}_1(Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \mathit{APreEven}_1(0, Y_{n-1}, X_{n-1}, Y_{n-2}) \uplus_{*} \mathit{Lpre}_1(Y_n, X_n) \\ \cup \\ B_{2n-3} \cap \mathit{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \uplus_{*} \mathit{Lpre}_1(Y_n, X_n) \\ \cup \\ \vdots \\ B_2 \cap \mathit{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \uplus_{*} \mathit{Lpre}_1(Y_n, X_n) \\ \cup \\ B_1 \cap \mathit{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \uplus_{*} \mathit{Lpre}_1(Y_n, X_n) \end{array}\right]$$

Proof. We first reformulate the algorithm for computing W in an equivalent form.

$$\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0.\mu X_0.
\begin{bmatrix}
T \\
B_{2n} \cap \text{Pre}_1(Y_{n-1}) \circledast\!\!\cup \text{Lpre}_1(W, X_n) \\
\cup \\
B_{2n-1} \cap \text{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \circledast\!\!\cup \text{Lpre}_1(W, X_n) \\
\cup \\
B_{2n-2} \cap \text{APreEven}_1(0, Y_{n-1}, X_{n-1}, Y_{n-2}) \circledast\!\!\cup \text{Lpre}_1(W, X_n) \\
\cup \\
B_{2n-3} \cap \text{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \circledast\!\!\cup \text{Lpre}_1(W, X_n) \\
\cup \\
\vdots \\
B_2 \cap \text{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \circledast\!\!\cup \text{Lpre}_1(W, X_n) \\
\cup \\
B_1 \cap \text{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \circledast\!\!\cup \text{Lpre}_1(W, X_n)
\end{bmatrix}$$

The reformulation is obtained as follows: since W is the fixpoint of Y_{n+1} we replace Y_{n+1} by W everywhere in the μ-calculus formula, and get rid of the outermost fixpoint. The above mu-calculus formula is a least fixpoint and thus computes W as an increasing sequence $T = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_m = W$ of states, where $m \geq 0$. Let $L_i = T_i \setminus T_{i-1}$ and the sequence is computed by computing T_i as follows, for $0 < i \leq m$:

$$\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0.\mu X_0.
\begin{bmatrix}
T \\
B_{2n} \cap \text{Pre}_1(Y_{n-1}) \circledast\!\!\cup \text{Lpre}_1(W, T_{i-1}) \\
\cup \\
B_{2n-1} \cap \text{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \circledast\!\!\cup \text{Lpre}_1(W, T_{i-1}) \\
\cup \\
B_{2n-2} \cap \text{APreEven}_1(0, Y_{n-1}, X_{n-1}, Y_{n-2}) \circledast\!\!\cup \text{Lpre}_1(W, T_{i-1}) \\
\cup \\
B_{2n-3} \cap \text{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \circledast\!\!\cup \text{Lpre}_1(W, T_{i-1}) \\
\cup \\
\vdots \\
B_2 \cap \text{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \circledast\!\!\cup \text{Lpre}_1(W, T_{i-1}) \\
\cup \\
B_1 \cap \text{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \circledast\!\!\cup \text{Lpre}_1(W, T_{i-1})
\end{bmatrix}$$

The above formula is obtained by simply replacing the variable X_n by T_{i-1}. The proof that $W \subseteq$ *Limit*$_1(IP, M, \text{Parity}(p) \cup \Diamond T)$ is based on an induction on the sequence $T = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_m = W$. For $1 \leq i \leq m$, let $V_i = W \setminus T_{m-i}$, so that V_1 consists of the last block of states that has been added, V_2 to the two last blocks, and so on until $V_m = W$. We prove by induction on $i \in \{1, \ldots, m\}$, from $i = 1$ to $i = m$, that for all $s \in V_i$, for all $\eta > 0$, there exists a memoryless strategy π_1^η for player 1 such that for all $\pi_2 \in \Pi_2$ we have

$$\text{Pr}_s^{\pi_1^\eta, \pi_2}\big(\Diamond T_{m-i} \cup \text{Parity}(p)\big) \geq 1 - \eta.$$

Since the base case is a simplified version of the induction step, we focus on the latter.

For $V_i \setminus V_{i-1}$ we analyze the predecessor operator that $s \in V_i \setminus V_{i-1}$ satisfies. The predecessor operators are essentially the predecessor operators of the almost-expression for case 1 modified by the addition of the operator $\text{Lpre}_1(W, T_{m-i}) \uplus$. Note that since we fix memoryless strategies for player 1, the analysis of counter-strategies for player 2 can be restricted to pure memoryless (as we have player-2 MDP). We fix the memoryless strategy for player 1 according to the witness distribution of the predecessor operators, and consider a pure memoryless counter-strategy for player 2. Let Q be the set of states where player 2 plays such the $\text{Lpre}_1(W, T_{m-i})$ part of the predecessor operator gets satisfied. Once we rule out the possibility of $\text{Lpre}_1(W, T_{m-i})$, then the μ-calculus expression simplifies to the almost-expression of case 2 with $Q \cup T$ as the set of target, i.e.,

$$\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0.\mu X_0. \left[\begin{array}{c} (T \cup Q) \\ B_{2n} \cap \text{Pre}_1(Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{APreEven}_1(0, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \cup \\ \vdots \\ B_2 \cap \text{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \end{array}\right]$$

This ensures that if we rule out $\text{Lpre}_1(W, T_{m-i})$ from the predecessor operators and treat the set Q as target, then by correctness of the almost-expression for case 2 we have that the $\text{Parity}(p) \cup \Diamond(Q \cup T)$ is satisfied with probability 1. By applying the Basic Lpre Principle (Lemma 12) with $Z = W$, $X = T_{m-i}$, $\mathcal{A} = \text{Parity}(p)$ and $Y = X \cup Q$, we obtain that for all $\eta > 0$ player 1 can ensure with a memoryless strategy that $\text{Parity}(p) \cup \Diamond T_{m-i}$ is satisfied with probability at least $1 - \eta$. This completes the inductive proof. With $i = m$ we obtain that for all $\eta > 0$, there exists a memoryless strategy π_1^η such that for all states $s \in V_m = W$ and for all π_2 we have $\text{Pr}_s^{\pi_1^\eta, \pi_2}(\Diamond T_0 \cup \text{Parity}(p)) \geq 1 - \eta$. Since $T_0 = T$, the desired result follows. ∎

We now define the dual predecessor operators (the duality will be shown in Lemma 15). We will first use the dual operators to characterize the complement of the set of limit-sure winning states for finite-memory infinite-precision strategies. We now introduce two fractional predecessor operators as follows:

$$\text{FrPreOdd}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$$
$$= \text{Fpre}_2(X_n, Y_n) \uplus \text{Apre}_2(X_n, Y_{n-1}) \uplus \cdots \uplus \text{Apre}_2(X_{n-i+1}, Y_{n-i}) \uplus \text{Pre}_2(X_{n-i})$$

$$\text{FrPreEven}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$$
$$= \text{Fpre}_2(X_n, Y_n) \uplus \text{Apre}_2(X_n, Y_{n-1})$$
$$\uplus \cdots \uplus \text{Apre}_2(X_{n-i+1}, Y_{n-i}) \uplus \text{Apre}_2(X_{n-i}, Y_{n-i-1})$$

The fractional operators are same as the PosPreOdd and PosPreEven operators, the difference is the $\text{Pospre}_2(Y_n)$ is replaced by $\text{Fpre}_2(X_n, Y_n)$.

Remark 2 *Observe that if we rule out the predicate Fpre*$_2(X_n, Y_n)$ *the predecessor operator FrPreOdd*$_2(i, Y_n, X_n, Y_n, \ldots, Y_{n-i}, X_{n-i})$ *(resp. FrPreEven*$_2(i, Y_n, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$*), then we obtain the simpler predecessor operator APreEven*$_2(i, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i})$ *(resp. APreOdd*$_2(i, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$*).*

The formal expanded definitions of the above operators are as follows:

$$\text{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) \circledast \text{Lpre}_1(Y_{n+1}, X_{n+1}) =$$

$$\left\{ s \in S \mid \forall \alpha > 0 \,.\, \exists \xi_1 \in \chi_1^s . \forall \xi_2 \in \chi_2^s . \begin{bmatrix} (P_s^{\xi_1,\xi_2}(X_{n+1}) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_{n+1})) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_n) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-i}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-i}) = 1) \end{bmatrix} \right\}.$$

$$\text{APreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) \circledast \text{Lpre}_1(Y_{n+1}, X_{n+1}) =$$

$$\left\{ s \in S \mid \forall \alpha > 0 \,.\, \exists \xi_1 \in \chi_1^s . \forall \xi_2 \in \chi_2^s . \begin{bmatrix} (P_s^{\xi_1,\xi_2}(X_{n+1}) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_{n+1})) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_n) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-i}) > 0 \wedge P_s^{\xi_1,\xi_2}(Y_{n-i}) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-i-1}) = 1) \end{bmatrix} \right\}.$$

The formal expanded definitions of the above operators are as follows:

$$\text{FrPreOdd}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) =$$

$$\left\{ s \in S \mid \exists \beta > 0. \forall \xi_1 \in \chi_1^s. \exists \xi_2 \in \chi_2^s. \begin{bmatrix} (P_s^{\xi_1,\xi_2}(Y_n) \geq \beta \cdot P_s^{\xi_1,\xi_2}(\neg X_n)) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-2}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-i}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_{n-i+1}) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(X_{n-i}) = 1) \end{bmatrix} \right\}.$$

$$\text{FrPreEven}_2(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) =$$

$$\left\{ s \in S \mid \exists \beta > 0. \forall \xi_1 \in \chi_1^s. \exists \xi_2 \in \chi_2^s. \begin{bmatrix} (P_s^{\xi_1,\xi_2}(Y_n) \geq \beta \cdot P_s^{\xi_1,\xi_2}(\neg X_n)) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_n) = 1) \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-2}) > 0 \wedge P_s^{\xi_1,\xi_2}(X_{n-1}) = 1) \\ \bigvee \\ \vdots \\ \bigvee \\ (P_s^{\xi_1,\xi_2}(Y_{n-i-1}) > 0 \wedge P_s^{\xi_1,\xi_2}(\neg X_{n-i}) = 1) \end{bmatrix} \right\}.$$

We now show the dual of Lemma 13.

Lemma 14 *For a parity function* $p : S \mapsto [1..2n]$ *we have* $Z \subseteq \neg \mathit{Limit}_1(\mathit{IP}, \mathit{FM}, \mathit{Parity}(p))$, *where* Z *is* *defined as follows:*

$$\mu Y_n.\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_1.\nu X_1.\mu Y_0.\nu X_0.\begin{bmatrix} B_{2n} \cap \mathit{FrPreEven}_2(0, Y_n, X_n, Y_{n-1} \\ \cup \\ B_{2n-1} \cap \mathit{FrPreOdd}_2(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \mathit{FrPreEven}_2(1, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \mathit{FrPreOdd}_2(2, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \cup \\ B_{2n-4} \cap \mathit{FrPreEven}_2(2, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}, Y_{n-3}) \\ \vdots \\ B_3 \cap \mathit{FrPreOdd}_2(n-1, Y_n, X_n, Y_{n-1}, X_{n-1}, \dots, Y_1, X_1) \\ \cup \\ B_2 \cap \mathit{FrPreEven}_2(n-1, Y_n, X_n, Y_{n-1}, X_{n-1}, \dots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \mathit{FrPreOdd}_2(n, Y_n, X_n, Y_{n-1}, X_{n-1}, \dots, Y_1, X_1, Y_0, X_0) \end{bmatrix}$$

Proof. For $k \geq 0$, let Z_k be the set of states of level k in the above μ-calculus expression. We will show that in Z_k, there exists constant $\beta_k > 0$, such that for every finite-memory strategy for player 1, player 2 can ensure that either Z_{k-1} is reached with probability at least β_k or else coParity(p) is satisfied with probability 1 by staying in $(Z_k \setminus Z_{k-1})$. Since $Z_0 = \emptyset$, it would follow by induction that $Z_k \cap \mathit{Limit}_1(\mathit{IP}, \mathit{FM}, \text{Parity}(p)) = \emptyset$ and the desired result will follow.

We obtain Z_k from Z_{k-1} by adding a set of states satisfying the following condition:

$$\nu X_n.\mu Y_{n-1}.\nu X_{n-1}.\cdots \mu Y_1.\nu X_1.\mu Y_0.\nu X_0.\begin{bmatrix} B_{2n} \cap \text{FrPreEven}_2(0, Z_{k-1}, X_n, Y_{n-1} \\ \cup \\ B_{2n-1} \cap \text{FrPreOdd}_2(1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{FrPreEven}_2(1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \cup \\ B_{2n-3} \cap \text{FrPreOdd}_2(2, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \\ \cup \\ B_{2n-4} \cap \text{FrPreEven}_2(2, Z_{k-1}, X_n, Y_{n-1}, X_{n-2}, Y_{n-2}, X_{n-2}, Y_{n-3}) \\ \vdots \\ B_3 \cap \text{FrPreOdd}_2(n-1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, \dots, Y_1, X_1) \\ \cup \\ B_2 \cap \text{FrPreEven}_2(n-1, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, \dots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{FrPreOdd}_2(n, Z_{k-1}, X_n, Y_{n-1}, X_{n-1}, \dots, Y_1, X_1, Y_0, X_0) \end{bmatrix}$$

The formula is obtained by removing the outer μ operator, and replacing Y_{n+1} by Z_{k-1} (i.e., we iteratively obtain the outer fixpoint of Y_{n+1}). If the probability of reaching to Z_{k-1} is not positive, then the following conditions hold:

- If the probability to reach Z_{k-1} is not positive, then the predicate $\text{Fpre}_2(X_n, Z_{k-1})$ vanishes from the predecessor operator $\text{FrPreOdd}_2(i, Z_{k-1}, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i})$, and thus the operator simplifies to the simpler predecessor operator $\text{APreEven}_2(i, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i})$.

- If the probability to reach Z_{k-1} is not positive, then the predicate $\text{Fpre}_2(X_n, Z_{k-1})$ vanishes from the predecessor operator $\text{FrPreEven}_2(i, Z_{k-1}, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$, and thus the operator simplifies to the simpler predecessor operator $\text{APreOdd}_2(i, X_n, Y_{n-1}, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$.

Hence either the probability to reach Z_{k-1} is positive, and if the probability to reach Z_{k-1} is not positive, then the above μ-calculus expression simplifies to

$$Z^* = \nu X_n.\mu Y_{m-1}\nu X_{m-1}\cdots\mu Y_1.\nu X_1.\mu Y_0. \begin{bmatrix} B_{2n} \cap \text{APreOdd}_2(0, X_n, Y_{n-1}) \\ \cup \\ B_{2n-1} \cap \text{APreEven}_2(1, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-2} \cap \text{APreOdd}_2(1, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_3 \cap \text{APreEven}_2(n-2, X_n, \ldots, Y_1, X_1) \\ \cup \\ B_2 \cap \text{APreOdd}_2(n-1, X_n, \ldots, Y_1, X_1, Y_0) \\ \cup \\ B_1 \cap \text{APreEven}_2(n-1, X_n, \ldots, Y_1, X_1, Y_0, X_0) \end{bmatrix}.$$

We now consider the parity function $p-1 : S \mapsto [0..2n-1]$, and observe that the above formula is same as the dual almost-expression for case 1. By correctness of the dual almost-expression we we have $Z^* \subseteq \{s \in S \mid \forall \pi_1 \in \Pi_1^M.\exists \pi_2 \in \Pi_2.\, \Pr_s^{\pi_1,\pi_2}(\text{coParity}(p)) = 1\}$ (since $\text{Parity}(p+1) = \text{coParity}(p)$). It follows that if probability to reach Z_{k-1} is not positive, then against every memoryless strategy for player 1, player 2 can fix a pure memoryless strategy to ensure that player 2 wins with probability 1. In other words, against every distribution of player 1, there is a counter-distribution for player 2 (to satisfy the respective APreEven_2 and APreOdd_2 operators) to ensure to win with probability 1. It follows that for every memoryless strategy for player 1, player 2 has a pure memoryless strategy to ensure that for every closed recurrent $C \subseteq Z^*$ we have $\min(p(C))$ is odd. It follows that for any finite-memory strategy for player 1 with $\mathcal{M}$, player 2 has a finite-memory strategy to ensure that for every closed recurrent set $C' \times \mathcal{M}' \subseteq Z^* \times \mathcal{M}$, the closed recurrent set C' is a union of closed recurrent sets C of Z^*, and hence $\min(p(C'))$ is odd (also see Example 3 as an illustration). It follows that against all finite-memory strategies, player 2 can ensure if the game stays in Z^*, then $\text{coParity}(p)$ is satisfied with probability 1. The Fpre_2 operator ensures that if Z^* is left and Z_{k-1} is reached, then the probability to reach Z_{k-1} is at least a positive fraction β_k of the probability to leave Z_k. In all cases it follows that $Z_k \subseteq \{s \in S \mid \exists \beta_k > 0.\forall \pi_1 \in \Pi_1^{FM}.\exists \pi_2 \in \Pi_2.\Pr_s^{\pi_1,\pi_2}(\text{coParity}(p) \cup \Diamond Z_{k-1}) \geq \beta_k\}$. Thus the desired result follows. ∎

Lemma 15 (Duality of limit predecessor operators). *The following assertions hold.*

1. *Given* $X_{n+1} \subseteq X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n \subseteq Y_{n+1}$, *we have*

$$\begin{array}{ll} \mathit{FrPreOdd}_2 & (i+1, \neg Y_{n+1}, \neg X_{n+1}, \neg Y_n, \neg X_n, \ldots, \neg Y_{n-i}, \neg X_{n-i}) \\ = & \neg(\mathit{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) \circledast\!\!\!\!\cup \mathit{Lpre}_1(Y_{n+1}, X_{n+1})). \end{array}$$

2. *Given* $X_{n+1} \subseteq X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i-1} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n \subseteq Y_{n+1}$ *and* $s \in S$, *we have*

$$\begin{array}{ll} \textit{FrPreEven}_2 & (i+1, \neg Y_{n+1}, \neg X_{n+1}, \neg Y_n, \neg X_n, \ldots, \neg Y_{n-i}, \neg X_{n-i}, \neg Y_{n-i-1}) \\ = & \neg(\textit{APreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) \uplus_* \textit{Lpre}_1(Y_{n+1}, X_{n+1})). \end{array}$$

Proof. We present the proof for part 1, and the proof for second part is analogous. To present the proof of the part 1, we present the proof for the case when $n = 1$ and $i = 1$. This proof already has all the ingredients of the general proof, and the generalization is straightforward as in Lemma 11.

Claim. We show that for $X_1 \subseteq X_0 \subseteq Y_0 \subseteq Y_1$ we have $\mathrm{Fpre}_2(\neg X_1, \neg Y_1) \uplus_* \mathrm{Apre}_2(\neg X_1, \neg Y_0) \uplus_* \mathrm{Pre}_2(\neg X_0) = \neg(\mathrm{Lpre}_1(Y_1, X_1) \uplus_* \mathrm{Apre}_1(Y_0, X_0))$. We start with a few notations. Let $\mathsf{St} \subseteq \Gamma_2(s)$ and $\mathsf{Wk} \subseteq \Gamma_2(s)$ be set of *strongly* and *weakly* covered actions for player 2. Given $\mathsf{St} \subseteq \mathsf{Wk} \subseteq \Gamma_2(s)$, we say that a set $U \subseteq \Gamma_1(s)$ satisfy *consistency* condition if

$$\begin{array}{l} \forall b \in \mathsf{St}.\ Dest(s, U, b) \cap X_1 \neq \emptyset \\ \forall b \in \mathsf{Wk}.\ (Dest(s, U, b) \cap X_1 \neq \emptyset) \vee (Dest(s, U, b) \subseteq Y_0 \wedge Dest(s, U, b) \cap X_0 \neq \emptyset) \end{array}$$

A triple $(U, \mathsf{St}, \mathsf{Wk})$ is consistent if U satisfies the consistency condition. We define a function f that takes as argument a triple $(U, \mathsf{St}, \mathsf{Wk})$ that is consistent, and returns three sets $f(U, \mathsf{St}, \mathsf{Wk}) = (U', \mathsf{St}', \mathsf{Wk}')$ satisfying the following conditions:

(1) $Dest(s, U', \Gamma_2(s) \setminus \mathsf{Wk}) \subseteq Y_1$;
(2) $\mathsf{St}' = \{b \in \Gamma_2(s) \mid Dest(s, U', b) \cap X_1 \neq \emptyset\}$
(3) $\mathsf{Wk}' = \{b \in \Gamma_2(s) \mid (Dest(s, U', b) \cap X_1 \neq \emptyset) \vee (Dest(s, U', b) \subseteq Y_0 \wedge Dest(s, U', b) \cap X_0 \neq \emptyset)\}$

We require that $(U, \mathsf{St}, \mathsf{Wk}) \subseteq (U', \mathsf{St}', \mathsf{Wk}')$ and also require f to return a larger set than the input arguments, if possible. We now consider a sequence of actions sets until a fixpoint is reached: $\mathsf{St}_{-1} = \mathsf{Wk}_{-1} = U_{-1} = \emptyset$ and for $i \geq 0$ we have $(U_i, \mathsf{St}_i, \mathsf{Wk}_i) = f(U_{i-1}, \mathsf{St}_{i-1}, \mathsf{Wk}_{i-1})$. Let $(U_*, \mathsf{St}_*, \mathsf{Wk}_*)$ be the set fixpoints (that is f cannot return any larger set). Observe that every time f is invoked it is ensured that the argument form a consistent triple. Observe that we have $\mathsf{St}_i \subseteq \mathsf{Wk}_i$ and hence $\mathsf{St}_* \subseteq \mathsf{Wk}_*$. We now show the following two claims.

1. We first show that if $\mathsf{Wk}_* = \Gamma_2(s)$, then $s \in \mathrm{Lpre}_1(Y_1, X_1) \uplus_* \mathrm{Apre}_1(Y_0, X_0)$. We first define the rank of actions: for an action $a \in U_*$ the rank $\ell(a)$ of the action is $\min_i a \in U_i$. For an action $b \in \Gamma_2(s)$, if $b \in \mathsf{St}_*$, then the strong rank $\ell_s(b)$ is defined as $\min_i b \in \mathsf{St}_i$; and for an action $b \in \mathsf{Wk}_*$, the weak rank $\ell_w(b)$ is defined as $\min_i b \in \mathsf{Wk}_i$. For $\varepsilon > 0$, consider a distribution that plays actions in U_i with probability proportional to ε^i. Consider an action b for player 2. We consider the following cases: (a) If $b \in \mathsf{St}_*$, then let $j = \ell_s(b)$. Then for all actions $a \in U_j$ we have $Dest(s, a, b) \subseteq Y_1$ and for some action $a \in U_j$ we have $Dest(s, a, b) \cap X_1 \neq \emptyset$, in other words, the probability to leave Y_1 is at most proportional to ε^{j+1} and the probability to goto X_1 is at least proportional to ε^j, and the ratio is ε. Since $\varepsilon > 0$ is arbitrary, the $\mathrm{Lpre}_1(Y_1, X_1)$ part can be ensured. (b) If $b \notin \mathsf{St}_*$, then let $j = \ell_w(b)$. Then for all $a \in U_*$ we have $Dest(s, a, b) \subseteq Y_0$ and there exists $a \in U_*$ such that $Dest(s, a, b) \cap X_0 \neq \emptyset$. It follows that in first case the condition for $\mathrm{Lpre}_1(Y_1, X_1)$ is satisfied, and in the second case the condition for $\mathrm{Apre}_1(Y_0, X_0)$ is satisfied. The desired result follows.

2. We now show that $\Gamma_2(s) \setminus \mathsf{Wk}_* \neq \emptyset$, then $s \in \mathrm{Fpre}_2(\neg X_1, \neg Y_1) \uplus_* \mathrm{Apre}_2(\neg X_1, \neg Y_0) \uplus_* \mathrm{Pre}_2(\neg X_0)$. Let $\overline{U} = \Gamma_1(s) \setminus U_*$, and let $B_k = \Gamma_2(s) \setminus \mathsf{Wk}_*$ and $B_s = \Gamma_2(s) \setminus \mathsf{St}_*$. We first present the required properties about the actions that follows from the fixpoint characterization.

(a) *Property 1.* For all $b \in B_k$, for all $a \in U_*$ we have

$$Dest(s,a,b) \subseteq \neg X_1 \wedge (Dest(s,a,b) \subseteq \neg X_0 \vee Dest(s,a,b) \cap \neg Y_0 \neq \emptyset).$$

Otherwise the action b would have been included in Wk_* and Wk_* could be enlarged.

(b) *Property 2.* For all $b \in B_s$ and for all $a \in U_*$ we have $Dest(s,a,b) \subseteq \neg X_1$. Otherwise b would have been included in St_* and St_* could be enlarged.

(c) *Property 3.* For all $a \in \overline{U}$, either

i. $Dest(s,a,B_k) \cap \neg Y_1 \neq \emptyset$; or

ii. for all $b \in B_s$, $Dest(s,a,b) \subseteq \neg X_1$ and for all $b \in B_k$,

$$Dest(s,a,b) \subseteq \neg X_1 \wedge (Dest(s,a,b) \subseteq \neg X_0 \vee Dest(s,a,b) \cap \neg Y_0 \neq \emptyset)$$

The property is proved as follows: if $Dest(s,a,B_k) \subseteq Y_1$ and for some $b \in B_s$ we have $Dest(s,a,b) \cap X_1 \neq \emptyset$, then a can be included in U_* and b can be included in St_*; if $Dest(s,a,B_k) \subseteq Y_1$ and for some $b \in B_k$ we have

$$(Dest(s,a,b) \cap X_1 \neq \emptyset) \vee (Dest(s,a,b) \cap X_0 \neq \emptyset \wedge Dest(s,a,b) \subseteq Y_0)$$

then a can be included in U_* and b can be included in Wk_*. This would contradict that $(U_*, \mathsf{St}_*, \mathsf{Wk}_*)$ is a fixpoint.

Let ξ_1 be a distribution for player 1. Let $Z = \text{Supp}(\xi_1)$. We consider the following cases to establish the result.

(a) We first consider the case when $Z \subseteq U_*$. We consider the counter distribution ξ_2 that plays all actions in B_k uniformly. Then by property 1 we have (i) $Dest(s,\xi_1,\xi_2) \subseteq \neg X_1$; and (ii) for all $a \in Z$ we have $Dest(s,a,\xi_2) \subseteq \neg X_0$ or $Dest(s,a,\xi_2) \cap \neg Y_0 \neq \emptyset$. If for all $a \in Z$ we have $Dest(s,a,\xi_2) \subseteq \neg X_0$, then $Dest(s,\xi_1,\xi_2) \subseteq \neg X_0$ and $\text{Pre}_2(\neg X_0)$ is satisfied. Otherwise we have $Dest(s,\xi_1,\xi_2) \subseteq \neg X_1$ and $Dest(s,\xi_1,\xi_2) \cap \neg Y_0 \neq \emptyset$, i.e., $\text{Apre}_2(\neg X_1, \neg Y_0)$ is satisfied.

(b) We now consider the case when $Z \cap \overline{U} \neq \emptyset$. Let $U_0 = U_*$, and we will iteratively compute sets $U_0 \subseteq U_i \subseteq Z$ such that (i) $Dest(s,U_i,B_s) \subseteq \neg X_1$ and (ii) for all $a \in U_i$ we have $Dest(s,a,B_k) \subseteq \neg X_0$ or $Dest(s,a,B_k) \subseteq \neg Y_0$ (unless we have already witnessed that player 2 can satisfy the predecessor operator). In base case the result holds by property 2. The argument of an iteration is as follows, and we use $\overline{U}_i = Z \setminus U_i$. Among the actions of $Z \cap \overline{U}_i$, let a^* be the action played with maximum probability. We have the following two cases.

i. If there exists $b \in B_s$ such that $Dest(s,a^*,b) \cap \neg Y_1 \neq \emptyset$, consider the counter action b. Since $b \in B_s$, by hypothesis we have $Dest(s,U_i,b) \subseteq \neg X_1$. Hence the probability to go out of $\neg X_1$ is at most the total probability of the actions in $Z \cap \overline{U}_i$ and for the maximum probability action $a^* \in Z \cap \overline{U}_i$ the set $\neg Y_1$ is reached. Let $\eta > 0$ be the minimum positive transition probability, then fraction of probability to go to $\neg Y_1$ as compared to go out of $\neg X_1$ is at least $\beta = \eta \cdot \frac{1}{|\Gamma_1(s)|} > 0$. Thus $\text{Fpre}_2(\neg X_1, \neg Y_1)$ can be ensured by playing b.

ii. Otherwise, by property 3, (i) either $Dest(s,a^*,B_k) \cap \neg Y_1 \neq \emptyset$, or (ii) for all $b \in B_s$ we have $Dest(s,a_*,b) \subseteq \neg X_1$ and for all $b \in B_k$

$$Dest(s,a^*,b) \subseteq \neg X_1 \wedge (Dest(s,a^*,b) \subseteq \neg X_0 \vee Dest(s,a^*,b) \cap \neg Y_0 \neq \emptyset)$$

If $Dest(s, a^*, B_k) \cap \neg Y_1 \neq \emptyset$, then chose the action $b \in B_k$ such that $Dest(s, a^*, b) \cap \neg Y_1 \neq \emptyset$. Since $b \in B_k \subseteq B_s$, and by hypothesis $Dest(s, U_i, B_s) \subseteq \neg X_1$, we have $Dest(s, U_i, b) \subseteq \neg X_1$. Thus we have a witness action b exactly as in the previous case, and like the proof above $\text{Fpre}_2(\neg X_1, \neg Y_1)$ can be ensured. If $Dest(s, a^*, B_k) \subseteq Y_1$, then we claim that $Dest(s, a^*, B_s) \subseteq \neg X_1$. The proof of the claim is as follows: if $Dest(s, a^*, B_k) \subseteq Y_1$ and $Dest(s, a^*, B_s) \cap X_1 \neq \emptyset$, then chose the action b^* from B_s such that $Dest(s, a^*, b^*) \cap X_1 \neq \emptyset$, and then we can include a^* to U_* and b^* to St_* (contradicting that they are the fixpoints). It follows that we can include $a^* \in U_{i+1}$ and continue.

Hence we have either already proved that player 2 can ensure the predecessor operator or $U_i = Z$ in the end. If U_i is Z in the end, then Z satisfies the property used in the previous cases of U_* (the proof of part a), and then as in the previous proof (of part a), the uniform distribution over B_k is a witness that player 2 can ensure $\text{Pre}_2(X_0) \circledast \text{Apre}_2(\neg X_1, \neg Y_0)$.

General case. The proof for the general case is a tedious extension of the result presented for $n = 1$ and $i = 1$. We present the details for the sake of completeness. We show that for $X_{n+1} \subseteq X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n \subseteq Y_{n+1}$, we have

$$\begin{array}{ll} \text{FrPreOdd}_2 & (i+1, \neg Y_{n+1}, \neg X_{n+1}, \neg Y_n, \neg X_n, \ldots, \neg Y_{n-i}, \neg X_{n-i}) \\ = & \neg(\text{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) \circledast \text{Lpre}_1(Y_{n+1}, X_{n+1})). \end{array}$$

We use notations similar to the special case. Let $\mathsf{St} \subseteq \Gamma_2(s)$ and $\mathsf{Wk} \subseteq \Gamma_2(s)$ be set of *strongly* and *weakly* covered actions for player 2. Given $\mathsf{St} \subseteq \mathsf{Wk} \subseteq \Gamma_2(s)$, we say that a set $U \subseteq \Gamma_1(s)$ satisfy *consistency* condition if

$$\begin{array}{l} \forall b \in \mathsf{St}.\ Dest(s, U, b) \cap X_{n+1} \neq \emptyset \\ \forall b \in \mathsf{Wk}.\ (Dest(s, U, b) \cap X_{n+1} \neq \emptyset) \vee \exists 0 \leq j \leq i.(Dest(s, U, b) \subseteq Y_{n-j} \wedge Dest(s, U, b) \cap X_{n-j} \neq \emptyset) \end{array}$$

A triple $(U, \mathsf{St}, \mathsf{Wk})$ is consistent if U satisfies the consistency condition. We define a function f that takes as argument a triple $(U, \mathsf{St}, \mathsf{Wk})$ that is consistent, and returns three sets $f(U, \mathsf{St}, \mathsf{Wk}) = (U', \mathsf{St}', \mathsf{Wk}')$ satisfying the following conditions:

$$\begin{array}{l} (1)\ Dest(s, U', \Gamma_2(s) \setminus \mathsf{Wk}) \subseteq Y_{n+1}; \\ (2)\ \mathsf{St}' = \{b \in \Gamma_2(s) \mid Dest(s, U', b) \cap X_{n+1} \neq \emptyset\} \\ (3)\ \mathsf{Wk}' = \{b \in \Gamma_2(s) \mid (Dest(s, U', b) \cap X_{n+1} \neq \emptyset) \vee \\ \qquad\qquad \exists 0 \leq j \leq i.(Dest(s, U', b) \subseteq Y_{n-j} \wedge Dest(s, U', b) \cap X_{n-j} \neq \emptyset)\} \end{array}$$

We require that $(U, \mathsf{St}, \mathsf{Wk}) \subseteq (U', \mathsf{St}', \mathsf{Wk}')$ and also require f to return a larger set than the input arguments, if possible. We now consider a sequence of actions sets until a fixpoint is reached: $\mathsf{St}_{-1} = \mathsf{Wk}_{-1} = U_{-1} = \emptyset$ and for $i \geq 0$ we have $(U_i, \mathsf{St}_i, \mathsf{Wk}_i) = f(U_{i-1}, \mathsf{St}_{i-1}, \mathsf{Wk}_{i-1})$. Let $(U_*, \mathsf{St}_*, \mathsf{Wk}_*)$ be the set fixpoints (that is f cannot return any larger set). Observe that every time f is invoked it is ensured that the argument form a consistent triple. Observe that we have $\mathsf{St}_i \subseteq \mathsf{Wk}_i$ and hence $\mathsf{St}_* \subseteq \mathsf{Wk}_*$. We now show the following two claims.

1. We first show that if $\mathsf{Wk}_* = \Gamma_2(s)$, then $s \in \text{Lpre}_1(Y_{n+1}, X_{n+1}) \circledast \text{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$. We first define the rank of actions: for an action $a \in U_*$ the rank $\ell(a)$ of the action is $\min_i a \in U_i$. For an action $b \in \Gamma_2(s)$, if $b \in \mathsf{St}_*$, then the strong rank $\ell_s(b)$ is defined as $\min_i b \in \mathsf{St}_i$; and for an action $b \in \mathsf{Wk}_*$, the weak rank $\ell_w(b)$ is defined as $\min_i b \in \mathsf{Wk}_i$. For $\varepsilon > 0$, consider a distribution

that plays actions in U_i with probability proportional to ε^i. Consider an action b for player 2. We consider the following cases: (a) If $b \in \mathsf{St}_*$, then let $j = \ell_s(b)$. Then for all actions $a \in U_j$ we have $Dest(s,a,b) \subseteq Y_{n+1}$ and for some action $a \in U_j$ we have $Dest(s,a,b) \cap X_{n+1} \neq \emptyset$, in other words, the probability to leave Y_{n+1} is at most proportional to ε^{j+1} and the probability to goto X_{n+1} is at least proportional to ε^j, and the ratio is ε. Since $\varepsilon > 0$ is arbitrary, the $\text{Lpre}_1(Y_{n+1}, X_{n+1})$ part can be ensured. (b) If $b \notin \mathsf{St}_*$, then let $j = \ell_w(b)$. Then for all $a \in U_*$ there exists $0 \leq j \leq i$ such that we have $Dest(s,a,b) \subseteq Y_{n-j}$ and there exists $a \in U_*$ such that $Dest(s,a,b) \cap X_{n-j} \neq \emptyset$. It follows that in first case the condition for $\text{Lpre}_1(Y_{n+1}, X_{n+1})$ is satisfied, and in the second case the condition for $\text{APreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$ is satisfied. The desired result follows.

2. We now show that $\Gamma_2(s) \setminus \mathsf{Wk}_* \neq \emptyset$, then

$$s \in \text{FrPreOdd}_2(i+1, \neg Y_{n+1}, \neg X_{n+1}, \neg Y_n, \neg X_n, \ldots, \neg Y_{n-i}, \neg X_{n-i}).$$

Let $\overline{U} = \Gamma_1(s) \setminus U_*$, and let $B_k = \Gamma_2(s) \setminus \mathsf{Wk}_*$ and $B_s = \Gamma_2(s) \setminus \mathsf{St}_*$. We first present the required properties about the actions that follows from the fixpoint characterization.

   (a) *Property 1.* For all $b \in B_k$, for all $a \in U_*$ we have

   $$Dest(s,a,b) \subseteq \neg X_{n+1} \wedge \exists 0 \leq j \leq i.(Dest(s,a,b) \subseteq \neg X_{n-j} \vee Dest(s,a,b) \cap \neg Y_{n-j} \neq \emptyset).$$

   Otherwise the action b would have been included in Wk_* and Wk_* could be enlarged.

   (b) *Property 2.* For all $b \in B_s$ and for all $a \in U_*$ we have $Dest(s,a,b) \subseteq \neg X_{n+1}$. Otherwise b would have been included in St_* and St_* could be enlarged.

   (c) *Property 3.* For all $a \in \overline{U}$, either

      i. $Dest(s,a,B_k) \cap \neg Y_{n+1} \neq \emptyset$; or

      ii. for all $b \in B_s$, $Dest(s,a,b) \subseteq \neg X_{n+1}$ and for all $b \in B_k$,

      $$Dest(s,a,b) \subseteq \neg X_{n+1} \wedge \exists 0 \leq j \leq i.(Dest(s,a,b) \subseteq \neg X_{n-j} \vee Dest(s,a,b) \cap \neg Y_{n-j} \neq \emptyset)$$

   The property is proved as follows: if $Dest(s,a,B_k) \subseteq Y_{n+1}$ and for some $b \in B_s$ we have $Dest(s,a,b) \cap X_{n+1} \neq \emptyset$, then a can be included in U_* and b can be included in St_*; if $Dest(s,a,B_k) \subseteq Y_{n+1}$ and for some $b \in B_k$ we have

   $$(Dest(s,a,b) \cap X_{n+1} \neq \emptyset) \vee \exists 0 \leq j \leq i.(Dest(s,a,b) \cap X_{n-j} \neq \emptyset \wedge Dest(s,a,b) \subseteq Y_{n-j})$$

   then a can be included in U_* and b can be included in Wk_*. This would contradict that $(U_*, \mathsf{St}_*, \mathsf{Wk}_*)$ is a fixpoint.

Let ξ_1 be a distribution for player 1. Let $Z = \text{Supp}(\xi_1)$. We consider the following cases to establish the result.

   (a) We first consider the case when $Z \subseteq U_*$. We consider the counter distribution ξ_2 that plays all actions in B_k uniformly. Then by property 1 we have (i) $Dest(s,\xi_1,\xi_2) \subseteq \neg X_{n+1}$; and (ii) for all $a \in Z$ there exists $j \leq i$ such that $Dest(s,a,\xi_2) \subseteq \neg X_{n-j}$ or $Dest(s,a,\xi_2) \cap \neg Y_{n-j} \neq \emptyset$. If for all $a \in Z$ we have $Dest(s,a,\xi_2) \subseteq \neg X_{n-i}$, then $Dest(s,\xi_1,\xi_2) \subseteq \neg X_{n-i}$ and $\text{Pre}_2(\neg X_{n-i})$ is satisfied. Otherwise, there must exists $j \leq i$ such that $Dest(s,\xi_1,\xi_2) \subseteq \neg X_{n+1-j}$ and $Dest(s,\xi_1,\xi_2) \cap \neg Y_{n-j} \neq \emptyset$, i.e., $\text{APreOdd}_2(i, \neg X_{n+1}, \neg Y_n \ldots, \neg X_{n-i+1}, \neg Y_{n-i})$ is satisfied.

(b) We now consider the case when $Z \cap \overline{U} \neq \emptyset$. Let $U_0 = U_*$, and we will iteratively compute sets $U_0 \subseteq U_\ell \subseteq Z$ such that (i) $Dest(s, U_\ell, B_s) \subseteq \neg X_{n+1}$ and (ii) for all $a \in U_\ell$ there exists $j \leq i$ such that $Dest(s, a, B_k) \subseteq \neg X_{n-j}$ or $Dest(s, a, B_k) \subseteq \neg Y_{n-j}$ (unless we have already witnessed that player 2 can satisfy the predecessor operator). In base case the result holds by property 2. The argument of an iteration is as follows, and we use $\overline{U}_\ell = Z \setminus U_\ell$. Among the actions of $Z \cap \overline{U}_\ell$, let a^* be the action played with maximum probability. We have the following two cases.

   i. If there exists $b \in B_s$ such that $Dest(s, a^*, b) \cap \neg Y_{n+1} \neq \emptyset$, consider the counter action b. Since $b \in B_s$, by hypothesis we have $Dest(s, U_\ell, b) \subseteq \neg X_{n+1}$. Hence the probability to go out of $\neg X_{n+1}$ is at most the total probability of the actions in $Z \cap \overline{U}_\ell$ and for the maximum probability action $a^* \in Z \cap \overline{U}_\ell$ the set $\neg Y_{n+1}$ is reached. Let $\eta > 0$ be the minimum positive transition probability, then fraction of probability to go to $\neg Y_{n+1}$ as compared to go out of $\neg X_{n+1}$ is at least $\beta = \eta \cdot \frac{1}{|\Gamma_1(s)|} > 0$. Thus $\text{Fpre}_2(\neg X_{n+1}, \neg Y_{n+1})$ can be ensured by playing b.

   ii. Otherwise, by property 3, (i) either $Dest(s, a^*, B_k) \cap \neg Y_{n+1} \neq \emptyset$, or (ii) for all $b \in B_s$ we have $Dest(s, a_*, b) \subseteq \neg X_{n+1}$ and for all $b \in B_k$

   $$Dest(s, a^*, b) \subseteq \neg X_{n+1} \wedge \exists 0 \leq j \leq i.(Dest(s, a^*, b) \subseteq \neg X_{n-j} \vee Dest(s, a^*, b) \cap \neg Y_{n-j} \neq \emptyset)$$

   If $Dest(s, a^*, B_k) \cap \neg Y_{n+1} \neq \emptyset$, then chose the action $b \in B_k$ such that $Dest(s, a^*, b) \cap \neg Y_{n+1} \neq \emptyset$. Since $b \in B_k \subseteq B_s$, and by hypothesis $Dest(s, U_\ell, B_s) \subseteq \neg X_1$, we have $Dest(s, U_\ell, b) \subseteq \neg X_{n+1}$. Thus we have a witness action b exactly as in the previous case, and like the proof above $\text{Fpre}_2(\neg X_{n+1}, \neg Y_{n+1})$ can be ensured. If $Dest(s, a^*, B_k) \subseteq Y_{n+1}$, then we claim that $Dest(s, a^*, B_s) \subseteq \neg X_{n+1}$. The proof of the claim is as follows: if $Dest(s, a^*, B_k) \subseteq Y_{n+1}$ and $Dest(s, a^*, B_s) \cap X_{n+1} \neq \emptyset$, then chose the action b^* from B_s such that $Dest(s, a^*, b^*) \cap X_{n+1} \neq \emptyset$, and then we can include a^* to U_* and b^* to St_* (contradicting that they are the fixpoints). It follows that we can include $a^* \in U_{\ell+1}$ and continue.

   Hence we have either already proved that player 2 can ensure the predecessor operator or $U_\ell = Z$ in the end. If U_ℓ is Z in the end, then Z satisfies the property used in the previous cases of U_* (the proof of part a), and then as in the previous proof (of part a), the uniform distribution over B_k is a witness that player 2 can ensure $\text{Pre}_2(\neg X_{n-i}) \uplus_* \text{APreOdd}_2(i, \neg X_{n+1}, \neg Y_n \ldots, \neg X_{n-i+1}, \neg Y_{n-i})$.

The desired result follows. ∎

Characterization of *Limit*$_1(IP, M, \Phi)$ set. From Lemma 13, Lemma 14, and the duality of predecessor operators (Lemma 15) we obtain the following result characterizing the limit-sure winning set for memoryless infinite-precision strategies for parity objectives.

Theorem 4 *For all concurrent game structures $\mathcal{G}$ over state space S, for all parity objectives $\Phi = Parity(p)$ for player 1, with $p : S \mapsto [1..2n]$, the following assertions hold.*

1. *We have Limit$_1(IP, M, \Phi)$ = Limit$_1(IP, FM, \Phi)$, and Limit$_1(IP, FM, \Phi) = W$, where W is defined as the μ-calculus formula in Fig 5, and $B_i = p^{-1}(i)$ is the set of states with priority i, for $i \in [1..2n]$.*

$$
\begin{array}{l}
\nu Y_n.\mu X_n.\nu Y_{n-1}.\mu X_{n-1}.\cdots \nu Y_1.\mu X_1.\nu Y_0.\mu X_0.\\
\left[\begin{array}{c}
B_{2n} \cap \text{Pre}_1(Y_{n-1}) \uplus_{*} \text{Lpre}_1(Y_n, X_n)\\
\cup\\
B_{2n-1} \cap \text{APreOdd}_1(0, Y_{n-1}, X_{n-1}) \uplus_{*} \text{Lpre}_1(Y_n, X_n)\\
\cup\\
B_{2n-2} \cap \text{APreEven}_1(0, Y_{n-1}, X_{n-1}, Y_{n-2}) \uplus_{*} \text{Lpre}_1(Y_n, X_n)\\
\cup\\
B_{2n-3} \cap \text{APreOdd}_1(1, Y_{n-1}, X_{n-1}, Y_{n-2}, X_{n-2}) \uplus_{*} \text{Lpre}_1(Y_n, X_n)\\
\cup\\
\vdots\\
B_2 \cap \text{APreEven}_1(n-2, Y_{n-1}, X_{n-1}, \ldots, Y_1, X_1, Y_0) \uplus_{*} \text{Lpre}_1(Y_n, X_n)\\
\cup\\
B_1 \cap \text{APreOdd}_1(n-1, Y_{n-1}, X_{n-1}, \ldots, Y_0, X_0) \uplus_{*} \text{Lpre}_1(Y_n, X_n)
\end{array}\right]
\end{array}
$$

Figure 5: μ-calculus formula for limit

2. *The set Limit*$_1(IP, FM, \Phi)$ *can be computed symbolically using the* μ*-calculus expression of Fig 5 in time* $\mathcal{O}(|S|^{2n+2} \cdot \sum_{s \in S} 2^{|\Gamma_1(s) \cup \Gamma_2(s)|})$.

3. *For* $s \in S$ *whether* $s \in$ *Limit*$_1(IP, FM, \Phi)$ *can be decided in NP* $\cap$ *coNP.*

The NP $\cap$ coNP bound follows directly from the μ-calculus expressions: the players can guess the ranking function of the μ-calculus formula and for each state the players guess the sequence of $(A_i, \mathsf{St}_i, \mathsf{Wk}_i)$ to witness that the predecessor operators are satisfied. The witnesses are polynomial and can be verified in polynomial time.

Independence from precise probabilities. Observe that the computation of all the predecessor operators only depends on the supports of the transition function, and does not depend on the precise transition probabilities. Hence the computation of the almost-sure and limit-sure winning sets is independent of the precise transition probabilities, and depends only on the supports. We formalize this in the following result.

Theorem 5 *Let* $\mathcal{G}_1 = (S, A, \Gamma_1, \Gamma_2, \delta_1)$ *and* $\mathcal{G}_2 = (S, A, \Gamma_1, \Gamma_2, \delta_2)$ *be two concurrent game structures that are equivalent, i.e.,* $\mathcal{G}_1 \equiv \mathcal{G}_2$*. Then for all parity objectives* Φ*, for all* $C_1 \in \{P, U, FP, IP\}$ *and* $C_2 \in \{M, FM, IM\}$ *we have (a) Almost*$_1^{\mathcal{G}_1}(C_1, C_2, \Phi) =$ *Almost*$_1^{\mathcal{G}_2}(C_1, C_2, \Phi)$*; and (b) Limit*$_1^{\mathcal{G}_1}(C_1, C_2, \Phi) =$ *Limit*$_1^{\mathcal{G}_2}(C_1, C_2, \Phi)$.

All cases of the above theorem, other than when $C_1 = IP$ and $C_2 = IM$ follows from our results, and the result for $C_1 = IP$ and $C_2 = IM$ follows from the results of [dAH00].

5 Conclusion

In this work we studied the bounded rationality problem for qualitative analysis in concurrent parity games, and presented a precise characterization. The theory of bounded rationality for quantitative analysis is future work, and we believe the results of this paper will be helpful in developing the theory.

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